AN OVERVIEW OF IMPROVED PROBABILISTIC METHODS FOR STRUCTURAL REDUNDANCY ASSESSMENT OF INDETERMINATE TRUSSES

BY

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ABSTRACT

The redundancy of a structure refers to the extent of strength that is not considered in design. For an indeterminate structure a member failure does not necessarily induces the loss of integrity or functionality of the structure; rather it will affect its potential for safely carrying any future load. Numerous methods have been introduced in structural reliability literature to measure and implement the redundancy in design. However, in accordance with the semi-probabilistic approach of the codes which aim to develop design method providing consistent level of redundancy within the structure, the probability of failure of the structure has been proposed and is widely used as a redundancy measure. A classical method to compute the probability of failure of the structure based on failure paths is presented as a reference in this thesis. However, although extensively used, this method has major shortcomings which may lead to a misrepresentation of the structure redundancy. By using a geometric representation of members' limit states associated with a loading regime, the research presented herein proposes an improved method for structural redundancy estimation that may be helpful to overcome problems associated with approximations and inconsistencies inherent in classics methods. Specific assumptions and/or procedures considered in the proposed method are described below.

(1) An approximation is given to make the events of member failures mutually exclusive.

(2) Geometric calculations are used to determine reliability indices and conditional reliability indices in order to establish closer bounds for the failure probability of individual structural members.

(3) System’s failure probability is obtained using the assumption and procedures outlined in (1) and (2) above.

(4) To further extend the method beyond geometrical redundancy, and to consider material redundancy, plasticity models commonly used in the structural analyses are considered in this study.

CHAPTER 1

INTRODUCTION

1.1 Objectives

The indeterminacy of a structure refers to the structure capacity to be stable after a member removal. The strength remaining in the stable damaged structure is called the reserve strength of the structure and is the measure of the structure redundancy. In the last thirty years numerous researches have been conducted leading to various different methods of the reserve strength quantification. Probabilistic methods are now preferred, because they provide a global description of the structure's redundancy and they can be implemented in semi-probabilistic codes. However, the research is still in its early stages of development; and the proposed methods (as described in this thesis) have many shortcomings which makes them inapplicable in a code. Indeed, so far, no code recommends a special approach for indeterminate redundant structures. The purpose of this study is to contribute to the improvement of these probabilistic methods by taking a classical method whose framework is widely used among researchers and improve upon its procedure.

1.2 Importance

The primary importance of measuring redundancy is to enable to design with a consistent level of risk within the structure. In a semi-probabilistic design every member is assigned with the same probability of failure. However, the consequences on the structure integrity or functionality of a member failure are different depending on the considered member. Instead of designing members with the same level of risk with regard to member failure, a measure of redundancy allows to design with the same level of risk with regard to the structure's failure.

The second importance of considering redundancy is the recognition that the unexpected can occur. This could be the occurrence of a rare event, the occurrence of poor structure characteristics or a construction fault. Additional safety can be provided to the structure by designing a system that can transfer load from one member to other members should one member break. This type of structure is called a fail-safe structure. On this type of structure the fracture is controlled and repairs are possible.

1.3 Scope

The evaluation of the redundancy described in this study consists of the following:

- A review of related work. Evaluation of the current practices and proposed method of considering indeterminacy and redundancy.
- A detailed presentation of a classical method to compute the probability of failure of an indeterminate structure. The study of this method enables us to outline its shortcomings.
- A proposed method based on the classical method framework. Strategies are proposed to overcome the classical method shortcomings.
- An application of the proposed method on a simple structure to show how it is implemented and the results found.

CHAPTER 2

REVIEW OF RELATED WORK

2.1 Definitions

Since some of the terms used in literature may have meanings that may not be clear to this study, the following definitions and nomenclature are used in this thesis for clarity.

2.1.1 Statically indeterminate structure.

2.1.1.1 Definition. The indeterminacy of a structure is a global characteristic, which corresponds to the lack of equilibrium equations compared with the number of unknown internal forces. Generally-speaking, a structural system is stable without a need to have redundancy. Therefore, redundant internal forces are not necessary to have a stable system; and such, they can be removed without deteriorating the stability of the system. In other words, some structural members can be totally removed without compromising the stability of the entire system. In a redundant system, however, a certain amount of damage can be tolerated without making the structure unstable. The more the system is indeterminate the more it can tolerate damage and removal of individual members due to any failure. However, it is emphasized that individual members may have different levels of influence on the overall integrity of the system. This is to say that, a system may geometrically appear to be indeterminate, however, for practical purposes, the same level of load may cause instability as soon as the member is failed. Thus, an indeterminate structure, even still stable after damage, may not be able to carry the same amount of load before damage.

2.1.1.2 Degree of static indeterminacy. A simple indicator of indeterminacy is to count the number of the unknown internal forces minus the total number of equations of equilibrium. This number is called the degree of indeterminacy. This can be demonstrated through the following equation:

$$dsi = (f \cdot m + r) - (e \cdot j + h)$$

Where all variables are non-negative integers as described below:

- f = forces = number of internal force in each member
- m = members = number of members
- r = restraints = number of restraints, i.e., boundary conditions
- e = equations = number of equilibrium equations per joint
- j = joints = number of joints
- h = hinges = number of hinges or other sections force releases

However this indicator is imperfect. Indeed, when applied to the entire structure, a substructure can present a deficit of unknowns whereas another substructure would present an excess of unknowns. In that case, the whole structure could be said to be indeterminate even if a mechanism exists inside the structure. Moreover depending of the geometry of the structure, by using this indicator a structure can be said to be stable, i.e., determinate or indeterminate whereas it is a mechanism (figure 2.1).

As a consequence, this indicator should be used with caution; and since the stiffness matrix of the entire system may be considered to be a more reliable indicator of the stability of a structure, the matrix may offer a better indicator of redundancy.

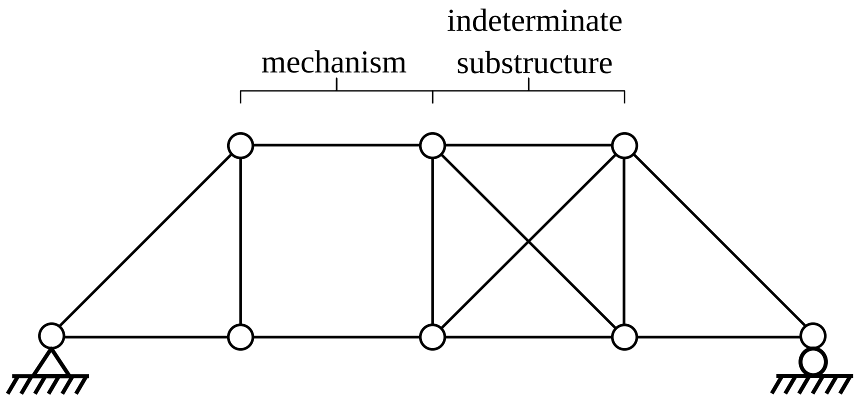


Figure 2.1 Truss structure with a mechanism and an indeterminate substructure

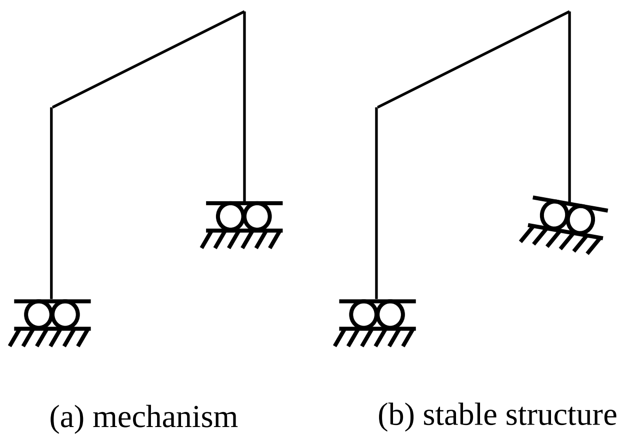
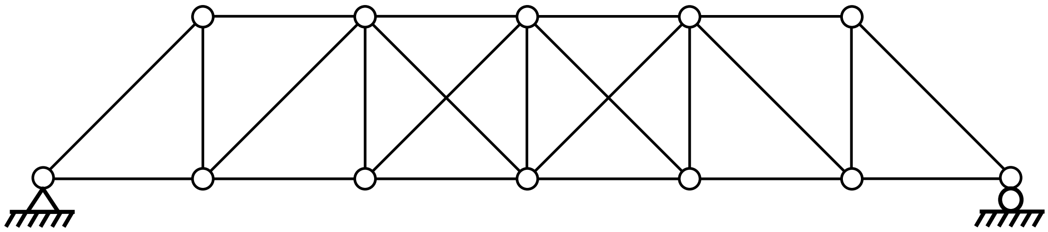
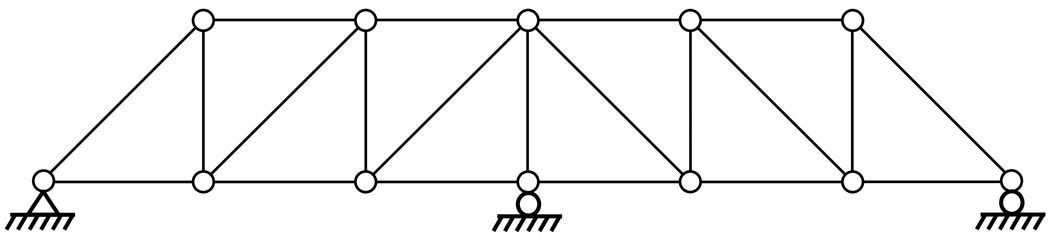


Figure 2.2 Comparison of two structures with one degree of redundancy

2.1.1.3 Internal and external degrees of statically indeterminacy. It is possible to divide degrees of indeterminacy into two categories – these are internal and external degrees of indeterminacy. The external degree of indeterminacy is the number of boundary conditions minus the number of global static equilibrium equations (3 for plane structures, 6 for space structures). The internal degree of indeterminacy refers to the number of unknown forces minus the total available equations of equilibrium (figure 2.3). Thereafter, in this thesis, only the internal degree of indeterminacy will be considered.



(a) internal indeterminacy



(b) external indeterminacy

Figure 2.3 Internal and external indeterminate structures

2.1.1.4 Degree of kinematic indeterminacy or degree of freedom. While the degree of static indeterminacy provides information about unknown member forces, the degree of kinematic indeterminacy (or degree of freedom) defines the number of unknown joint displacements and rotations. This is to say that the degree of static indeterminacy is the size of the flexibility matrix in the force method and the degree of kinematic indeterminacy is the size of the stiffness matrix in the stiffness method. The degree of kinematic indeterminacy is easier to determine; and the stiffness method can be implemented (in computer assisted structural analyses) in a straightforward manner to arrive at equations for solving the degrees of freedom.

2.1.2 Fail-Safe structure. A fail-safe condition refers to a type of structure, which in the event of failure will respond in a way that will cause no harm to the users or occupants. A proper structural system design is intended to prevent or mitigate unsafe consequences of a failure and to protect the users or occupants. For example, in a structural system, an event of safe failure is the local failure of a single member; whereas, the unsafe consequence may be corresponding to the case when the entire structure collapses.

2.1.3 Load path. A load path is a set of forces inside the structure members capable of carrying the load from the top of the structure to the bottom (see Fig. 2.4). A load path depends on the type of structure and on the load. The load path, which takes place in the original intact structure, is called the principal load path. In case of an indeterminate structure, alternate load paths may exist in a damaged structure. Those load paths may provide conditions of "over-strength" to the structure.

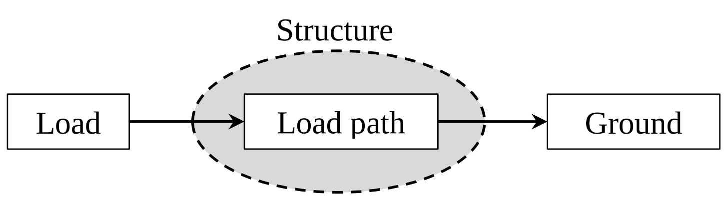


Figure 2.4 Load path

2.1.4 Material and geometrical redundancy.

2.1.4.1 Material redundancy. The material redundancy of a structure is the additional resistance a member (that is considered failed for practical purposes) can provide to the system. This type of redundancy is provided by the rheology of the material. There are two types of material redundancy: the displacement-redundancy, the constraint-redundancy.

The displacement-redundancy enables a failed element of the structure to accommodate additional displacement whereas the constraint-redundancy enables the failed element to resist to additional constraint. In general, elements are both displacement-redundant and constraint-redundant.

The stress-strain constitutive relation of a material describes the material redundancy. The following diagrams (figure 2.5) provide idealized comprehensive examples for different types of redundancy.

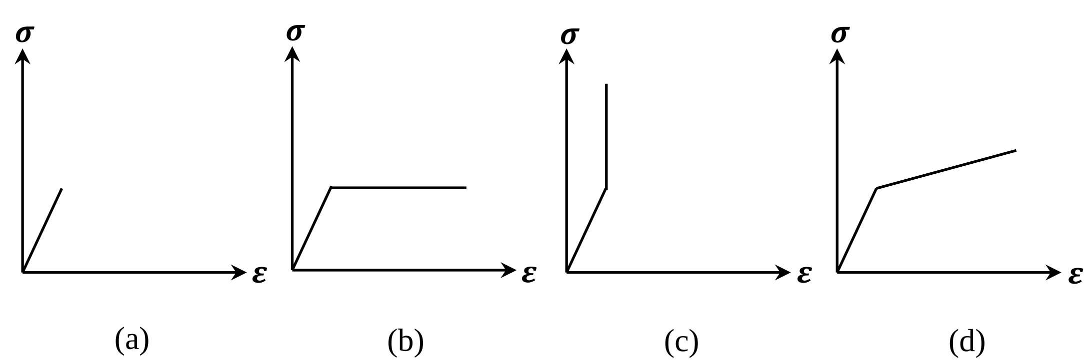


Figure 2.5 Stress-strain constitutive relations

The situation denoted by (a) in Figure 2.5 has no redundancy. The situation denoted by (b) has only a displacement redundancy. The situation denoted by (c) has only a constraint redundancy. The situation denoted by (d) has both displacement and constraint redundancy.

2.1.4.2 Geometric redundancy. Geometric redundancy is the ability of a structure, for which all members are considered without material redundancy, to redistribute the load among its members when damage to certain members has occurred. Since redundancy is generally associated with indeterminacy the indeterminacy of at least some parts of the structure is required to provide alternative load paths. However the degree of static indeterminacy is not a consistent measure for geometric redundancy. A structure is said to be geometrically redundant if it possesses at least a few members that are considered geometrically redundant.

2.1.4.3 System redundancy. A system is said to be redundant if it has some members which are either geometrically redundant or materially redundant. As described earlier, material redundancy can be interpreted as the reserve strength of a member itself, whereas geometric redundancy can be taken as the reserve strength of a structure beyond point of yield. This distinction between material and geometric redundancy is only theoretical since both occur in a given structure indistinctively. Indeed, when a member is considered as failed and starts being materially redundant, its behavior will be modified. This behavior modification leads to a reorganization of forces in the structure; and as such, alternative load paths are formed to sustain the loads. At this stage, the geometric redundancy is then used. Thus the redundancy of a system is the reserve strength available from both material redundancy and geometrical redundancy of member for preventing failure of an entire structural system upon a failure of a single element.

2.1.5 Failure path. A failure path is a sequence of single member failure leading to the development of a mechanism in the structure. The length of the failure path, i.e. the number of member failure in the sequence is directly related to the local degrees of indeterminacy. The set of all possible failure paths provides for a complete description of the structure's failure state. Figure 2.6 illustrates potential failure paths in a simple truss system.

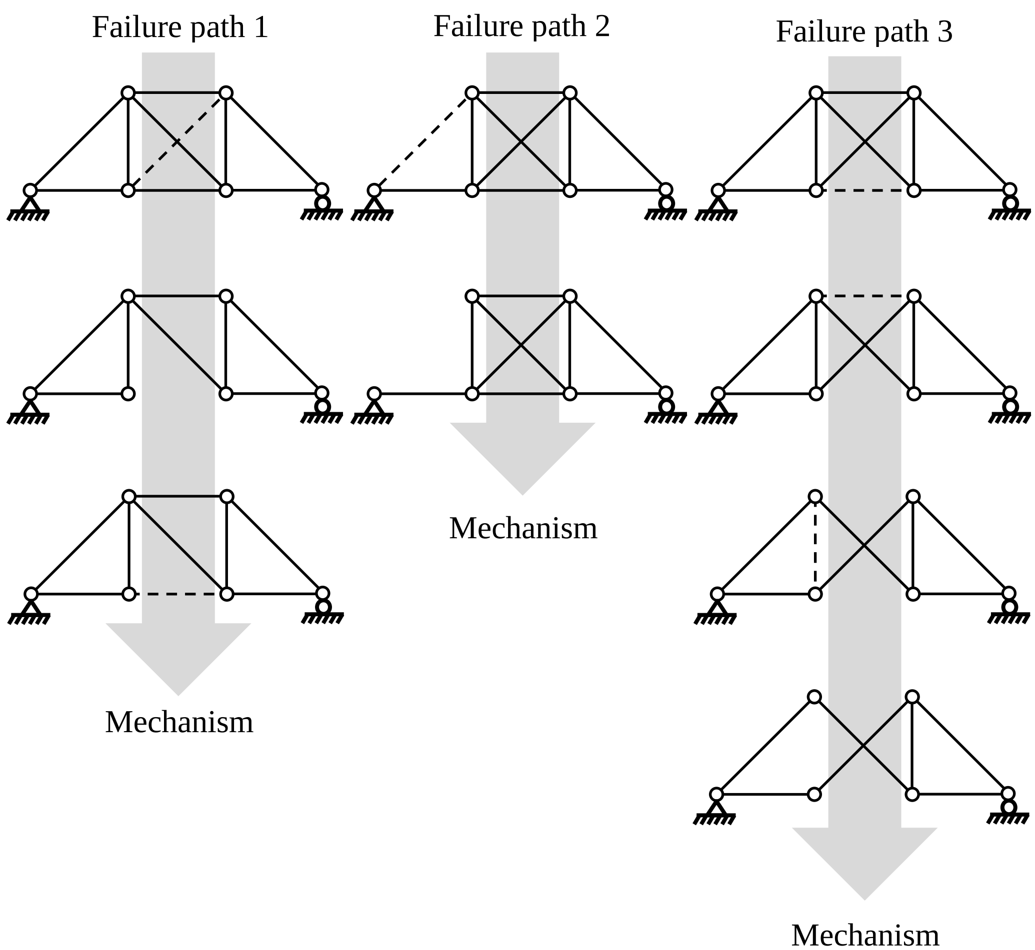


Figure 2.6 Failure paths

As seen in figure 2.6, three possible failure paths are present in the structure. It can be seen that different failure paths have different lengths in terms of the number of members that need to fail before the formation of a mechanism in the structure. The three presented failure paths are a non-exhaustive example of the variety of the possible failure paths. To represent possible failure paths, a tree diagram can be used as shown in figure 2.7.

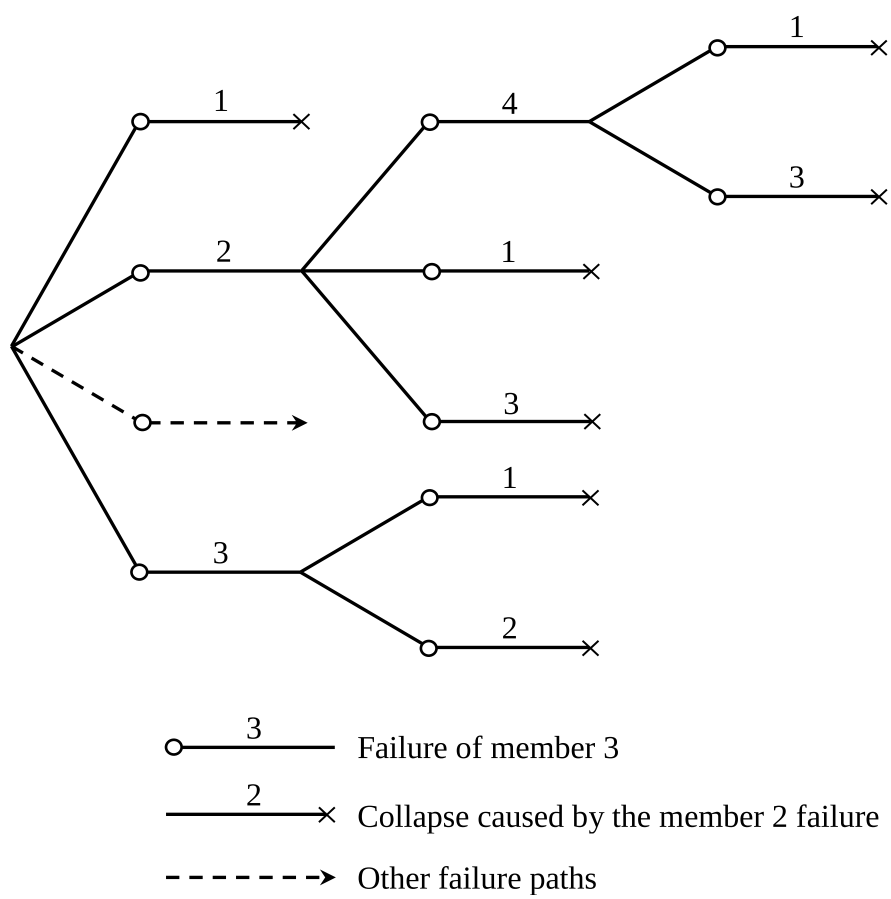


Figure 2.7 Failure paths tree

As a complete representation of all the possible fracture mechanisms, a failure paths tree enables us to take into account the role of every elements in every given failure scenario. However, for complex structures, the failure paths tree may not be readily evident. In these cases, algorithms can be used to generate every failure path with an aid of a computer program.

2.1.6 The reserve strength. The reserve strength can be linked to the redundancy. As described earlier, the redundancy is defined as the structure's capacity in sustaining more loads after an initial failure. The amount of strength remaining in the structure is called the reserve strength. Thus, in simple terms, the reserve strength can be considers as a measure of the redundancy of the structure. However, more definite measures to quantify the reserve strength are used as described below.

- Possible ways to quantify the reserve strength with a deterministic approach:

  First a member must be chosen as a candidate to fail. Then a structural analysis on the damaged structure is conducted. The over strength can be chosen freely as a characteristic of the damaged structure. For example, one can choose the maximum load for a given limit deflection, or, one can choose the maximum load before the failure of the next member. Thus, in the case of a deterministic approach, the reserve strength can only be a partial measure given a specific situation

- Possible ways to quantify the reserve strength with a probabilistic approach: The over-strength of a damaged structure is introduced via a probability value. This value represents the chance of failure and is in fact equal to the probability that in a specific member the over-strength value is exceeded. For a structural system, this probability value is the probability that the system over-strength is exceeded or a failure mechanism has taken place. Since this probability can be computed at any stage when members are failing, the difference between the probability values from one loading stage to another (as structural degradation takes place) can be used as a measure for the reserve capacity as discussed later in this thesis.

2.2 Importance of redundancy

2.2.1 Structure behavior during failure. In current practice, the strength evaluation of a structure is typically based on using elastic analysis to determine the distribution of load effects in the members and then checking the ultimate section capacity of those members against the applied load effects. However, when a failure criterion is used to define the failure of the entire structure (rather than the failure of a single member), this strength evaluation may not be accurate. Material redundancy, i.e. ductility, of the components permits local yielding (especially in most heavily loaded members) and, coupled with geometric redundancy, enables subsequent redistribution of the internal forces. As a result, a system can continue to carry additional loading even after one member has yielded, which has conventionally been adopted as the failure criterion in structural strength evaluation. This means that a structure with redundancy has additional reserve strength such that the failure of one element does not result in the failure of the complete system. The reserve strength is a measure of the redundancy of a system and its capacity to safely redistribute forces among its members when damage has occurred. When designing without taking into account the redundancy, one needs to provide for additional safety using stronger structural members. This of course will result in additional costs. Designing with redundancy potentially enables the engineer to control the fracture process that leads to failure. This ability of controlling the fracture process is governed by two main concepts, namely, the reserve strength and the discretization of the failure process. As explained later, through a probabilistic formulation, these two concepts can be considered in structural analysis to provide an improved measure describing the system redundancy. More specifically, these two concepts can be introduced through a "damage curve" [10]. A damage curve is a relationship between a normalized damage level and an applied load that produces the damage (see Fig. 2.8). The damage level 0 represents no damage, i.e., no yielding of any member, and 1.0 represents total collapse of the structure. For cases where the damage is partial, the damage is proportional to the number of failed members.

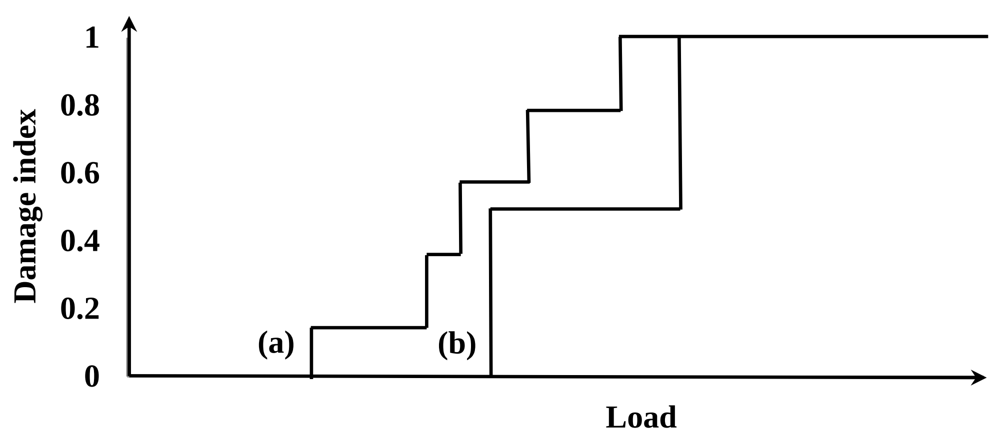


Figure 2.8 Damage curve example

Figure 2.8 presents two damage curves of two redundant systems. Structure (a) spreads out the failure process in five steps; whereas the structure (b) does it in only two steps. System (b) has more reserve strength, i.e., can carry more load, than does structure (a). Although these two curves provide a description of the discretization of the failure process and the reserve strength, without further analysis, it will be difficult to determine the inherent level of safety of each structure through redundancy of which system. A probabilistic definition of the reserves strength (including the discretization of the failure process) will be helpful to introduce the problem through probable values of reserved capacity and thus offer an alternate solution in defining the redundancy and its significance in defining safety.

To control the fracture process at the design stage, one must know the loads and its variation in time. Unexpected loads can occur and, as a result, the reserved strength of the structure may be needed to resist against them. By incorporating an accurate estimate for redundancy, and designing for a fail-safe condition, the fracture process is controlled leading to a design with a more consistent level of risk.

2.2.2 Representation of the risk. Designing with redundancy and with an accepted risk level for collapse is not necessarily leading to a structure that is stronger, but rather the design will have the capability to account for the fact that an unexpected load can occur and still a safe level of performance is attained. This could be the occurrence of rare events such as floods or earthquakes, poor maintenance, human error, unexpected fatigue cracking, a severe wind storm, etc. By incorporating an accurate estimate of redundancy, the engineer is able to reduce the damaging effects of occurrences of these rare events and have a more consistent level of accepted risk.

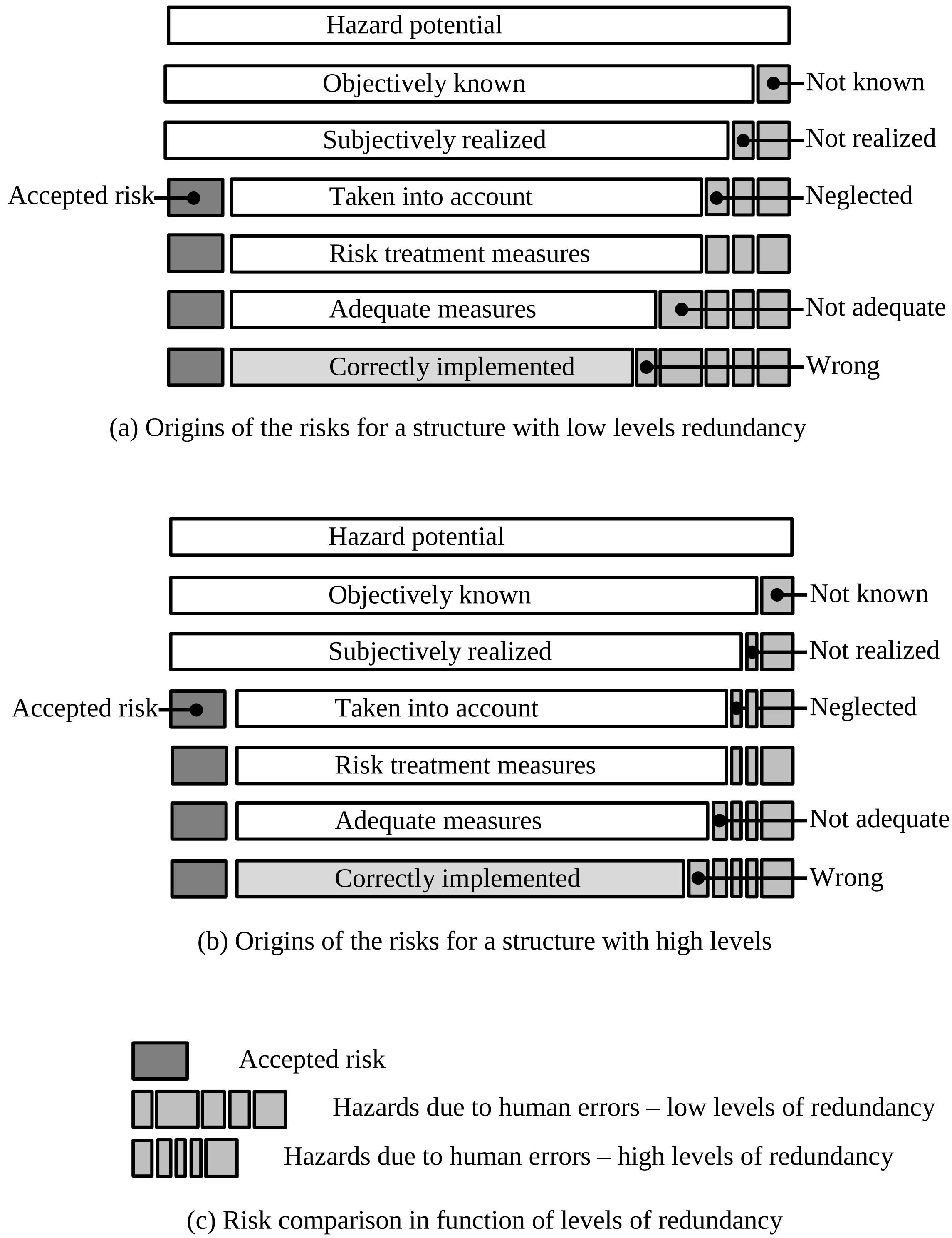


Figure 2.9 Origins of the risks

Figure 2.9 presents diagrams that depict the relation between risk and redundancy for a given system. The significance of low and high levels of redundancy on the risk of failure is demonstrated. A structure with a low level of redundancy, for example, is designed without recognizing that unexpected loads or service conditions may occur. As a consequence the hazards due to the unexpected events are relatively severe as oppose to a structure designed with redundancy. On the other hand, a structure designed with a high level of redundancy has more consistent level of risk because the accepted risk, which is decided by the engineer and the structure is accommodated for the risk, is closer to the risk from hazards due to the unexpected events.

2.3 Criterion for a realistic evaluation of redundancy

Several criteria have been proposed to assess the overall redundancy of a system. Some relevant studies will be introduced in this section. Then, the criterion that will be used in the thesis will be introduced.

2.3.1 The deterministic redundancy factor and the redundancy index. The following discussion is from research by Frangopol and Curley [7]. The deterministic redundancy factor and the redundancy index consider the effect of damage or failure of a given component on the system strength in the damaged condition. The structural residual strength factor r is defined as follows:

$$r = \frac{R_{res}}{R_{ult}}$$

In which,

- R_{ult} is the ultimate strength of the undamaged system

- R_{res} is the residual strength of the damaged system

The structural redundancy factor R is defined as follows:

$$R = \frac{R_{ult}}{R_{ult} - R_{res}}$$

This redundancy factor ranges from a value of 1.0 when the damaged structure has no reserve strength to a value of infinity when the structural damage has no influence on the reserve strength of the system. The redundancy factor R is calculated for each component. Based on the redundancy factor of various members, the engineer can identify the important members as those with having the greatest influence on the reserved strength of the structure.

However, this criterion can only be used as a partial description of redundancy. Indeed, this factor has some drawbacks:

- Every member must be computed independently. It does not provide a global description of the structure.
- The loading must be the same type for every computed member. It does not enable the engineer to provide a complete description of the loading.
- The total redundancy of the structure is not considered. The damaged structured is considered to be failed when a second member fails: and as such, the representation of the redundancy is not total but at best partial.

2.3.3 The beta index. A probabilistic measure of the redundancy of a structure based on the structural redundancy factor proposed Frangopol and Curley [7] is used in this thesis. According to this model, a parameter (referred to as the reliability index) is introduced as described below.

$$\beta(R) = \frac{\beta(intact\ structure)}{\beta(intact\ structure) - \ \beta(damaged\ structure)}$$

In which,

- $\beta(intact\ structure)$ is the beta index or reliability index of the intact structure
- $\beta(damaged\ structure)$ is the reliability index of the damaged structure

The beta index (or reliability index) is the input in the cumulative distribution function of the standardized normalized distribution. This simple indicates that:

$$\beta = \Phi^{-1}(1 - P_f)$$

In which P_f is the probability of failure (or damage) of the structure. This index can also be expressed as the difference between the resistance of the structure and the applied load with the variance of this difference taken as unity. This probabilistic measure of redundancy $\beta(R)$ varies between 1 and infinity. Unlike the structural redundancy factor R, $\beta(R)$ enables us to fully describe the variety in loading. However there are still two main drawbacks in arriving at an accurate value for the structural redundancy factor R. These are (1) the fact that still the redundancy for each member must be computed separately; and (2) that the system redundancy so obtained is not accurately representing the total system redundancy.

2.3.3 The successive damage probability. The successive damage probability is also used to develop a method for redundancy as proposed by Mohammadi, Longinow and Suen [11]. This method determines the effectiveness of redundant components in a structural system as the determining factor in defining redundancy. Specific application presented is for bridges. The method requires the calculation of the probability of damage to the structural system before and after removal of a redundant component. Damage is defined as the failure of one or more components; and the probability of damage of the system is:

$$P(D) = P(\bigcup_{i=1}^{m} E_i)$$

In which:

- E_i is the event of failure of the individual component i

$P(D)$ is calculated for the intact structure and then again for the system with one component failed. The values for $P(D)$ are then compared at each stage after a partial damge; and, if the results are nearly equal, the structure is considered to have a moderate degree of redundancy. Continuing with this process, then a second component, in addition to the first component, is considered failed and a new probability of damage is calculated. This process continues until the damage probability shows a sudden increase. This increase indicates that the collapse of the system has become certain. The number of components removed to result in the collapse stage is a measure of the redundancy of the system. If, however, $P(D)$ shows a significant increase after removal of a first single member, the system is considered to have no redundancy. A criterion that would limit the increase of the system probability of damage could be established to provide uniformity to the method. This approach is a good attempt in providing a method that does not present the drawbacks of the two first methods. However, this method does not, to some extent, represent the very accurate estimate of the redundancy, since the process which leads from the first damaged member to the complete failure of the structure is arbitrary chosen. Indeed, using this method, the redundancy of a structure depends on the sequence of member removals. Some members may appear to be more (or less) critical depending on the failure sequence in which they are involved. Using this method, the probability formulation is only in computation of the risk of damage and not in the sequence of member failures leading to the system collapse. A more realistic method would have provided for a probabilistic approach in the sequence of member failures also.

2.3.4 The probability of failure. In light of the fact that the redundancy computation using the aforementioned method requires the computation of the probability of failure, the following provides a brief review of the relevant topics in the theory of probability.

- The reliability criterion must be global. It must consider, as much as possible, an accurate geometry and/or characteristic for the structure and the applied loads
- The reliability criterion must be straightforward limiting the need for the engineer to rely on personal judgment.
- The reliability criterion must also be consistent with the safety of the structure; such that it can be used for ranking structures from those with lesser to those with higher safety levels.

- The reliability criterion must be also present a unique representation of the safety of the system. This means, using the chosen reliability criterion, the analysis must always result in the same probability of failure.

With regards to these requirements, the rationale for a realistic evaluation of redundancy will be based on the probability of failure of the structure given the structure has already been damaged. To evaluate this probability, a probability space and three subspace of this space must be defined, as described below:

- The load and structure space. This space includes all the necessary characteristics to define both a structure (in terms of geometry and component resistances) and its loading.
- The subspace of consistent load and structure. This subspace includes every component of the load and the structure's configuration for which the load and the structure represents a realistic structural behavior. For example, the subspace will only include a lateral loading on a frame; and the structure's configuration includes lateral load resisting components of the frame (either bracing or moment connections).
- The subspace of damaged structure. This subspace includes every component from the subspace of consistent load and structure for which the load damages the structure.
- The subspace of failed structure. This subspace includes every component from the subspace of damaged structure for which the load has completely failed the structure (see figure b2.10).

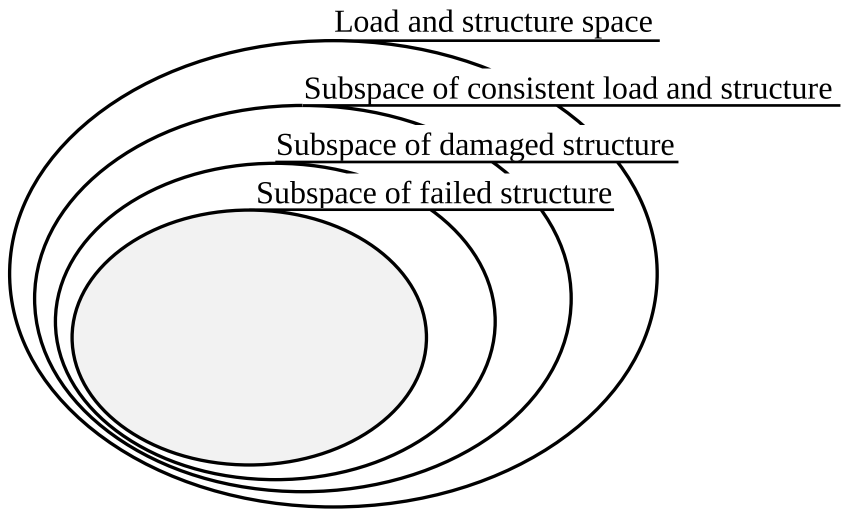


Figure 2.10 Venn diagram of the load and structure space

The criterion for the evaluation of redundancy is the probability of failure given the structure has already been damaged which is the probability of occurrence of a specific scenario from the subspace of the failed structure, when considering only the subspace of damaged structure.

2.4 Present ways of considering redundancy in design codes

Redundancy, as the capacity of a structure to provide reserved strength and to withstand additional loads, regardless the condition of the structure, is a well-known concept among civil engineers. However, in most application, the redundancy is not often considered. This is because most codes do not require any redundancy evaluation, despite the fact that it has been recognized that taking redundancy into account can lead to a more consistent level of safety for the system. In most cases, the code-recommended procedures call for checking the structure's conditions at two levels; namely, the section level check (to ensure safety through an allowable stress or load/resistance factors), and the element level check (to ensure stability and/or safety through an allowable deflection, an allowable P-delta value or acceptable buckling deformation). And the reason why the redundancy is left out from this code-recommended process can be summarized as follows:

- Redundancy is difficult to evaluate. Researchers do not agree on a common and straightforward method on how to measure redundancy.
- Redundancy measures proposed in the literature are incompatible with the present way of structural design which focuses on single members rather than the system assemblage.

Recognizing these two main issues, new directions in certain advanced codes are toward implementing redundancy in design. These codes, first, define redundancy and introduce a measure of redundancy. Then based on the measured redundancy, they recommend modification factors that can be used to consider redundancy while preserving a classical design process, i.e. to design at the member level.

2.4.1 Redundancy as provisioned in AASHTO.

2.4.1.1 Introductory remarks. Under the design requirements for bridges given by the Standard Specifications for Highway Bridges, 17th Edition [1], the member resistance is based on, among other parameters, a system factor which modifies the design resistance of a member with regard to the behavior of the structure. However, limited guidance is provided in the code on how this can be done. To fill this gap, an official report has been published by the National Cooperative Highway Research Program [6].

The NCHRP Report 406, on Redundancy in Highway Bridge Superstructures [6], provides a framework for considering redundancy in design and in the load capacity evaluation of highway bridge structures. The report starts by recognizing that the structural components of a bridge do not behave independently but interact with each other to form a structural system, which is generally ignored by bridge specifications that deal with individual components. The report attempts to bridge the gap between a component-by-component design and a system design.

2.4.1.2 Framework. In order to be able to consider redundancy, the report introduces four limit states that provide an extended definition for safety.

- The member failure limit state. This is the traditional check of individual member safety using elastic analysis and nominal member capacity.
- The ultimate limit state. This is defined as the ultimate capacity of the bridge system. It corresponds to the formation of a collapse mechanism.
- The functionality limit state. This is defined as a maximum acceptable live load displacement.
- The damaged conditions limit state. This is defined as the ultimate capacity of the bridge system after damage to one main load-carrying element.

Thus, the level of system safety is not only provided by the member failure limit state but also other limit states that may occur. These four limit states are represented by a load factor, LF_1 for the member failure limit state, LF_u for the ultimate limit state, LF_f for the functionality limit state and LF_d for the damaged condition limit state. These load factors correspond to the factors by which the weights of two side-by-side AASTHO HS-20 trucks located at the mid-span of the bridge are multiplied before the limit state is reached. The load factor values are calculated from an incremental structural analysis using the finite element method considering the elastic and inelastic material behavior of the structure. The factors provide for a margin of safety to ensure that highway bridges will possess a minimum level of safety whether intact or upon a component failure. Based on the NCHRP 406, the bridge is said to have adequate levels of redundancy if the following conditions are satisfied:

- The system's reserved ultimate capacity is defined via a parameter, $R_u = \frac{LF_u}{LF_1}$, which is to be greater than or equal to 1.30.
- The system reserved functionality limit is defined via a parameter, $R_f = \frac{LF_f}{LF_1}$, which is to be greater than or equal to 1.10.
- The system reserved damaged condition limit is defined via a parameter, $R_d = \frac{LF_d}{LF_1}$, which is to be greater than or equal to 0.50.

Bridge designs that do not satisfy the criteria given above are not considered to have a sufficient level of redundancy and should be strengthened. On the other hand, bridges that satisfy the above criteria are permitted to have less conservative member designs.

The next in the framework proposed in NCHRP 406 is the introduction of a redundancy factor ϕ_{red} defined as:

$$\phi_{red} = \min\left(r_1 \cdot r_u, r_1 \cdot r_f, r_1 \cdot r_d\right)$$

Where:

- r_1 is the member reserved ration defined as $r_1 = \frac{LF_1}{LF_{1req}}$ where LF_1 is given by the incremental structural analysis; whereas LF_{1req} is given by the AASTHO specifications. Bridge members that are designed to exactly match AASTHO specifications will produce a member reserved ratio of 1.0.
- r_u is the redundancy ratio for the ultimate limit state defined as $r_u = \frac{R_u}{1.30}$.
- r_f is the redundancy ratio for the functionality limit state defined as $r_f = \frac{R_f}{1.10}$.
- r_d is the redundancy ratio for the damaged condition limit state defined as $r_d = \frac{R_d}{0.50}$.

The redundancy factor must be compared to 1.0. If ϕ_{red} is less than 1.0, it indicates that the bridge under consideration has an inadequate level of system redundancy. A redundancy factor greater than 1.0 indicates a sufficient level of redundancy. The redundancy factor can be used as a penalty-reward factor whereby non redundant bridges would be required to have higher capacities; whereas redundant bridge would be permitted to have lower member resistances.

The report concludes by giving a user-friendly system factor to be used to adjust the member capacities when designing single members. This system factor ϕ_s is intended to provide the right amount of redundancy, i.e. a redundancy factor equals to 1.0 for design. The purpose of the system factor is to make engineers able to use a traditional design procedure while considering the redundancy of the system. The system factor has to be computed for each type of structure and is tabulated for the most common ones. The equation to be used in design is to incorporate this factor as a measure of capacity reduction as explained below:

$$\phi_s \cdot \phi \cdot R = \gamma_d \cdot D_n + \gamma_l \cdot L_n \cdot (1 + I)$$

Where:

- γ_d and γ_l are respectively the dead load and live load factor.
- D_n and L_n are the dead load effect and the live load effect on the considered member.
- I is the dynamic impact factor.
- ϕ is the member resistance factor.
- ϕ_s is the system factor accounting for redundancy effects.

When ϕ_s is equal to 1.0 the equation becomes the same as the standard design equation. The system factor tables are calibrated such that a system factor equal to 1.0 indicates that the exact amount of redundancy is incorporated in design.

2.4.1.3 Concluding remarks. The NCHRP Report 406 intends to introduce a framework to consider the redundancy in bridge structures. To do so, the authors provide, for the most common bridge structures, tables prescribing a modification factor to be used in the design equation in order to take the redundancy into account. For these cases, using this system factor, the engineer is capable of considering the redundancy without any additional calculations. The redundancy, which is a system property, is taking into account at the member level. For the non-tabulated cases, NCHRP 406 provides a framework consisting of an incremental step-by-step structural analysis (from initial loading until the total collapse of the bridge) that can be used in the computation of a redundancy factor. This factor can be used as a penalty-reward factor in order to redesign the bridge with an adjusted geometry.

However, the proposed framework is a simplified procedure and may lead to misperception of the redundancy when applied to complex structures and loadings. Therefore, the incremental structural analysis prescribed in the framework considers only one type of loading consisting of two side-by-side AASTHO HS-20 trucks located at the mid-span of the bridge. The lack of variety of the load may lead to a misrepresentation of the governing load for members in a complex structural system. As a consequence, a bridge which is sensitive to failure to another type of loading than the one recommended in the framework, would have its redundancy over-estimated. Without a complete representation of the variety of the loadings, the method should not be used for structures with complex geometry and/or behavior. Moreover, another weakness of the method in case of complex structures is the necessity of choosing damage scenarios for computing the damage condition limit state. The method assumes that worst damage scenarios are known in order to perform a limited number of incremental structural analyses. However, for complex structures, a full understanding of the collapse mechanisms is difficult to reach. As a result, the number of incremental structural analyses needed to describe every possible behavior of the structure will rise with the complexity of the structure and may not be feasible in an everyday design situation.

2.4.2 AISC provisions. Under the requirements for buildings subject to earthquake loads given by the 2006 IBC and ASCE 7-05, the earthquake load is based on a redundancy factor ρ to take into account the additional safety provided by alternative load paths in case of failure. The load combinations to be used with earthquakes are:

$$(1.2 + 0.2 \cdot S_{DS}) \cdot D + \rho \cdot Q_E + L + 0.2 \cdot S$$

$$(0.9 - 0.2 \cdot S_{DS}) \cdot D + \rho \cdot Q_E$$

Where:

- ρ is the redundancy factor. ρ is either equal to 1.0 or 1.3.
- Q_E is the horizontal effect of the seismic load.

The redundancy factor is introduced as a multiplier of the horizontal effect of the seismic load. It is equal to 1.0 if the loss or removal of any component would not result in more than 33 percent reduction in the story strength for any story resisting more than 35 percent of the base shear. Other cases where the redundancy factor can be taken as 1.0 are for: (1) the structures assigned to SDC B or SDC C, (2) the drift and P-delta calculations, (3) the design of non-structural components, (4) cases when over-strength is required, and (5) cases when systems with passive energy devices are considered. For any other case, the redundancy factor is 1.3.

The redundancy factor highlights the need of redundancy in structures when designing for earthquake. The earthquake design requires the structure to withstand large inelastic deformations. Thus, the designed members are closer in terms of expected constraints and deformations to failure than in a conventional design. As a consequence, failure is likely to happen requiring the need for fail-safe structures. The redundancy factor acts more like penalty-reward factor which intends to provide consistent levels of safety. To do so, structures with adequate levels of redundancy are rewarded with a low factor of redundancy which will enable them to be designed for low earthquake loads. This will result in weaker members. However, structures with inadequate levels of redundancy are penalized with a high factor of redundancy which will constraint them to be designed for high earthquakes loads. This will result in stronger members.

Contrary to the AASHTO code, the AISC procedure uses the redundancy factor as a load multiplier and not a resistance multiplier. Furthermore, the AISC code requires redundancy for earthquakes design only.

Notwithstanding the fact that the AISC procedure provides for a rapid framework to evaluate and take into account the redundancy in earthquake design, the description of the redundancy of the structure (in question) is not very accurate and may in fact result in inconsistency with regards to the inherent level of system safety. The main defect of the AISC procedure is that it is not able to provide a consistent measure of redundancy. And in fact, using the procedure, the result for the structure will be either redundant or non-redundant. The lack of information concerning the degree of redundancy of the structure leads to an over-simplified design measure resulting in an exaggerated differentiation between the structure which is almost redundant in terms of the definition and conditions given by the AISC code and the structure which is barely redundant. In fact, although the two structures may have close levels of safety, the first one will have to use a redundancy factor of 1.3 whereas the second one will be able to use a redundancy factor of 1.0.

2.4.3 Provisions in the Eurocodes. The fundamental requirements given in the first Eurocode (EN-1990) for the system reliability include:

- Serviceability requirement: the structure during its intended life, with appropriate degrees of reliability and in an economic way, will remain fit for the use for which it was intended to.
- Safety requirement: the structure will sustain all actions and influences likely to occur during its use.
- Fire requirement: the structural resistance shall be adequate for the required period of fire exposure.
- Robustness requirement: the structure will not be damaged by events such as explosion, impact or consequences of human errors, to an extent disproportionate to the original cause.

The redundancy is taken into account in the last requirement concerning the robustness of a structure. The term used is slightly different between the American and European code. In the European code, the robustness is the structure's capacity to limit the extent of damage; whereas in the American code, an interpretation for robustness may refer to the extent of degradation that the structure can suffer without losing some of its intended functionality. For instance, in the American practice, the robustness of a beam, when considering serviceability against the live load defection, can be determined through the largest reduction in the cross section that can be made and still satisfy the maximum allowable live load deflection. Roughly speaking, the European code refers to redundancy by using the term robustness, which of course has a different interpretation in the American literature. Because of the definition used by the European literature, in the following discussion, we will use the terms redundancy and robustness interchangeably.

The redundancy or robustness of a structure is one of the four main requirements of the Eurocode. The robustness of a structure appears to be a major design concern. However, no methods are provided to evaluate and take it into account. Moreover, in practice, it is left to the owner to decide whether or not require the design engineer to consider robustness in his/her design. Currently, a committee has been mandated to provide more detailed provisions concerning robustness, which is expected to be released in the coming years.

CHAPTER 3

PRESENTATION OF A CLASSICAL METHOD TO COMPUTE THE PROBABILITY OF FAILURE OF A STRUCTURE

3.1 Existing methods

It was seen in the section 2.3.4 that a criterion for the evaluation of redundancy can be the probability of failure of a structure given that the structure has already been damaged once. It was said that this probability can be calculated as being the probability of occurrence of an object from the subspace of failed structure when considering the subspace of damage structure as the whole probability space. To do so, a mathematical model must be constructed. This mathematical model must include uncertainties in applied loads, materials, real kind of member connections etc. and must define a probabilistic analysis of structure. In this regard by using reliability concepts, failure probability of structures can be determined. First damage should be defined in order to delineate the set of damaged structure into which the failure probability of failure of structure will be calculated.

Considerable progress has been made in recent years in the reliability estimation of structures under uncertain loads. Many algorithms have been developed and successfully implemented to estimate the reliability of a structure at the element and system level. These methods can be categorized into three groups, namely:

- Numerical integration method
- Failure path method
- Simulation

The numerical integration methods are exact and only possess approximation because of the numerical integration process. They are analytical methods mixed with a numeric solution. The random loads and random resistances of the structure are compared owing to the limit state equations (which are needed in order to define first the set of damaged structure). In this set further calculations are made to describe every failed structure possibilities. The numerical integration methods consider every failure paths. For very simple structures, the close-form of the integration can be maintained. For moderately complex structures, the integration must be numerically approximated. For highly complex structures, due to the number of limit states to consider, and the number of possible failure paths, numerical integration methods turn out to be time consuming and should avoided. Shortcuts in the processes must be found to make the algorithms faster.

In the failure path methods such as the branch and bound method [15] and the truncated method [11], failure paths with low probability are bounded and dominant failure paths are chosen. Then upper and lower bounds are determined for system failure probability. There are two obstacles on these methods. Firstly, there are so many failure paths in large structures and the required processes would be time-consuming. Secondly, the obtained upper and lower bounds of the failure probability may not be narrow.

The Monte Carlo simulation method has often been used to calculate the probability of failure and to verify the results of other reliability analysis methods. In this method, the random loads and random resistance of a structure are simulated in order to know if the structure fails or not according to the limit states. The probability of failure is the relative ratio between the number of failure occurrences and the total number of simulations. This approach is easy to employ but when encountered low failure probability, which often happens in real structural systems, the number of simulation becomes large such that the method becomes unpractical for most realistic problems.

Although the analytical methods provide an elegant approach for handling the system reliability problems, one of the short comings of them is that for real structures numerous failure paths need to be considered. All efforts and simplifying assumptions made in the three types of methods previously introduced denote the importance of computational time.

3.2 Structural Analysis for trusses

A truss is defined as tree-dimensional framework of straight prismatic members connected at their ends by frictionless hinged joints. The members of a truss are subjected to axial compressive or tensile forces only. Hereafter, the matrix stiffness method will be presented. This method of analysis is general, in the sense that it can be applied to any structure regardless of its degree of indeterminacy. The second main advantage of the use of the matrix stiffness method is that it can be simply implemented in a computer algorithm.

3.2.1 Global and local coordinate systems. Two types of coordinate systems are employed to specify the structural and loading data and to establish the necessary force-displacement relations. These are referred as the global (or structural) and the local (or member) coordinate systems.

The global system coordinate system describes the overall geometry and the load-deformations relationships for the entire structure. The global coordinate used in the text is a right-handed XYZ coordinate (figure 3.1).

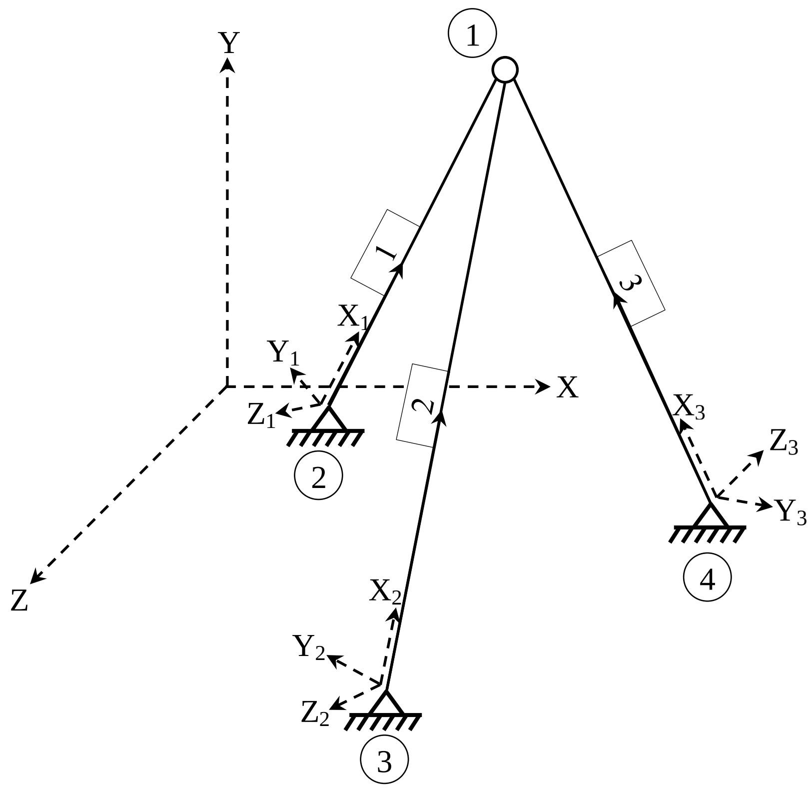


Figure 3.1 Global and local coordinate systems

The local system is convenient to derive the member force-displacement relationship in terms of the forces and displacements in the directions along and perpendicular to members. The local systems are indexed to a number corresponding to the member number. For the purpose of the study, the node must be indexed by numbers too.

3.2.2 Degrees of freedom, joint load vector and reaction vector. The degrees of freedom of a structure are defined as the independent joints displacement that must be specified to describe the deformed of a structure subjected to a loading. In the case of a truss, these joints displacement are only translations. All the joint displacements can be collectively written in a matrix form which is called the joint displacement vector.

$$\boldsymbol{d} = \begin{bmatrix} d_1 \\ d_2 \\ \cdots \\ d_n \end{bmatrix}$$

The joint displacement vector is written in the global coordinate system. The number of degrees of freedom can be obtained by subtracting the number of restrained displacements due to the supports from the number of joint displacements of the unsupported structure which is three times the number of joints.

For trusses, forces are only applied at the joint locations and can be divided into two categories: the external loading forces and the reaction forces. In general, the external loading can be applied at the location and in the direction of every degree of freedom. Hereafter the external loading forces will be simply called the joint loads. These joints loads can be indexed and collectively written in a matrix form which is called the joint load vector.

$$\boldsymbol{P} = \begin{bmatrix} P_1 \\ P_2 \\ \cdots \\ P_n \end{bmatrix}$$

The reaction forces correspond to the forces developed into the structure due to the restrained displacement. These reactions can be indexed as well and written using a matric form which is called the reaction vector.

$$\boldsymbol{R} = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{bmatrix}$$

It should be noted that the joint load vector and the displacement vector have the same number of components. The reaction vector number of components corresponds to the difference between the number of joint displacements of the unsupported structure and the number of components of the joint load vector.

Figure 3.2(a) depicts the deformed shape of a truss and the joint displacement vector for individual joint. The joint displacement vector is given in the global coordinate system and its components are indexed from the joint number one to the last joint and respectively to X, Y and Z. Figure 3.2(b) shows the joint load vector and the reaction vector indexation for individual joint. The same numbering process is chosen, i.e. from the first joint to the last beginning with the X component through the Z component. To conclude, the joint displacement vector, the joint load and the reaction vector for the entire structure are computed by assembling the individual indexed vectors.

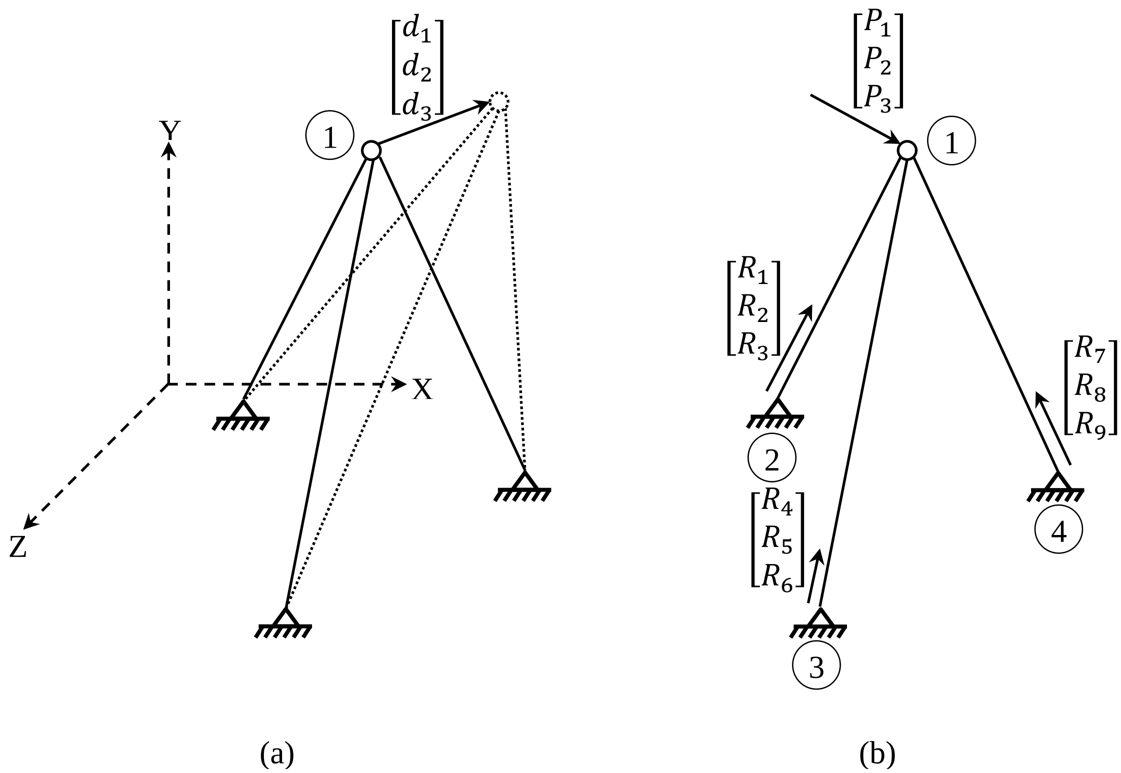


Figure 3.2 Degrees of freedom, joint load vector and reaction vector

3.2.3 Member stiffness in the local coordinate system. For a single member, the displacements at the ends and the forces at those ends are related to each other, in the local coordinate system, by the stiffness matrix. A member has six degrees of freedom or end displacements; however since displacements which are perpendicular to the member do not induce any force, for the purpose of analysis only the two displacements along the member are considered.

The local end displacement vector $\boldsymbol{u_m}$ for a member m is expressed as:

$$\boldsymbol{u_m} = \begin{bmatrix} u_{m1} \\ u_{m2} \end{bmatrix}_{local\ m}$$

In the same way, the end forces vector corresponding to $\boldsymbol{Q_m}$ is defined as follows:

$$\boldsymbol{Q_m} = \begin{bmatrix} Q_{m1} \\ Q_{m2} \end{bmatrix}_{local\ m}$$

The relationship between the local end forces $\boldsymbol{Q_m}$ and the end displacements $\boldsymbol{u_m}$, for the member space trusses, is written as:

$$\boldsymbol{Q_m} = \boldsymbol{k_m} \cdot \boldsymbol{u_m}$$

$$\begin{bmatrix} Q_{m1} \\ Q_{m2} \end{bmatrix}_{local\ m} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}_{local\ m} \cdot \begin{bmatrix} u_{m1} \\ u_{m2} \end{bmatrix}_{local\ m}$$

Where:

- $\boldsymbol{k_m}$ is the member stiffness matrix
- E is the member young's modulus
- A is the member section
- L is the member length

Figure 3.3(a) below is an illustration of the deformed shape of a single member m which shows the locations of the vectors corresponding to the applied forces and to the displacements. The vector components are written in the local coordinate system corresponding the member m.

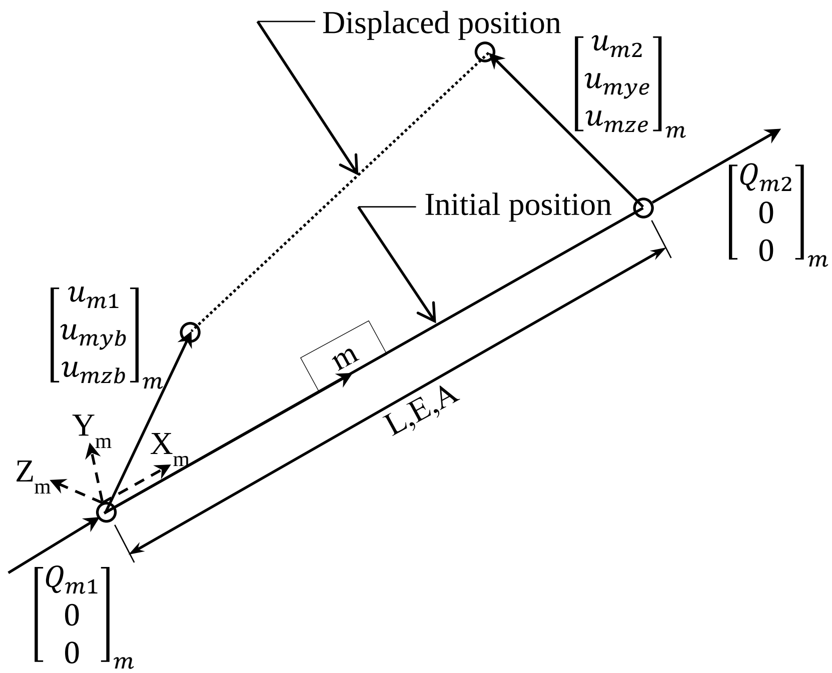


Figure 3.3 Member forces and displacement in the local coordinate system

3.2.4 Coordinate transformations. The relationship in the local coordinate member system between the local end forces $\boldsymbol{Q_m}$ and the end displacement $\boldsymbol{u_m}$ for space trusses has been explicated in the previous section. However this relationship as being related to a particular coordinate system cannot be used further. This member relationship must be computed at a global level. The equivalent of the member end displacements $\boldsymbol{u_m}$ and the end forces $\boldsymbol{Q_m}$ in the local coordinate system are the end displacements $\boldsymbol{v_m}$ and the end forces $\boldsymbol{F_m}$ in the global coordinate system. Figure 3.4 depicts the geometric relationship between the local and global member end displacements and the end forces.

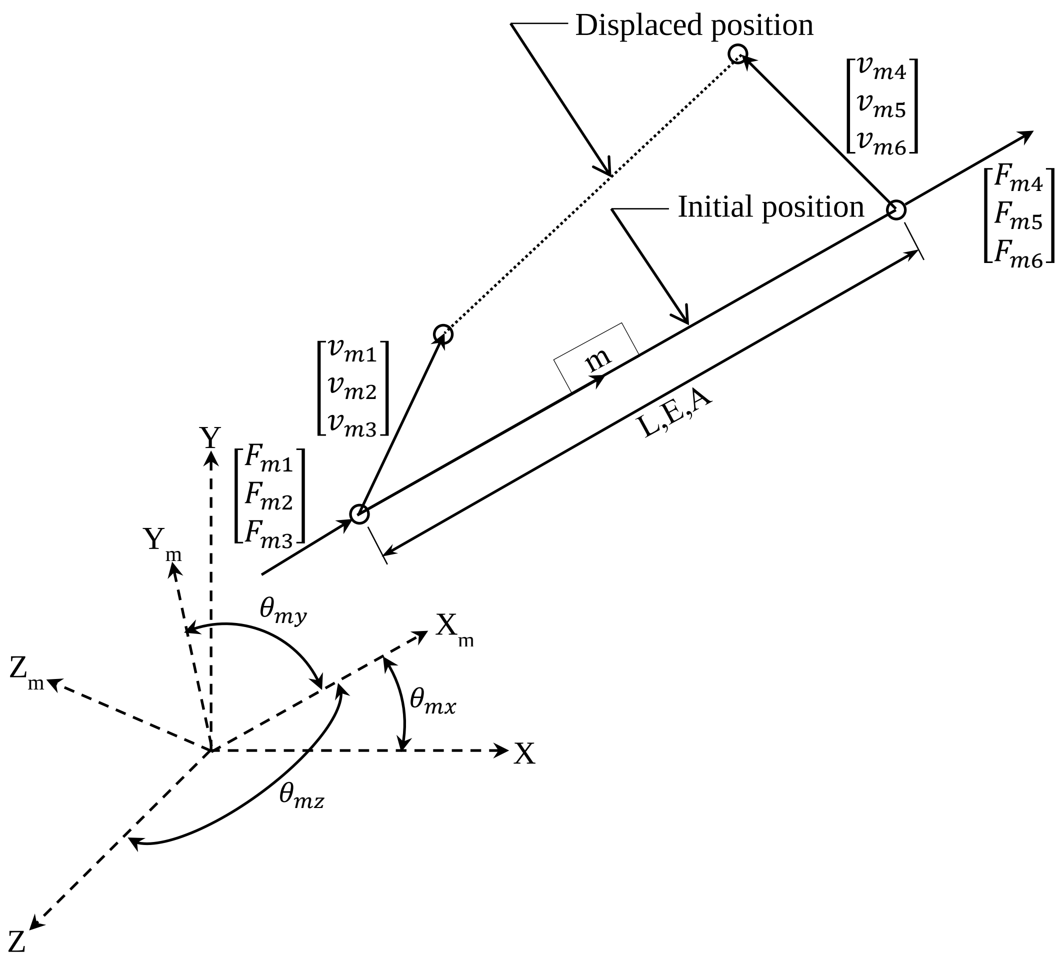


Figure 3.4 Member forces and displacement in the global system and relationship between the local and global coordinate system

The relationship between the local and global coordinates for the member forces and displacements can be written in terms of angles θ_x, θ_y and θ_z as follows:

$$\begin{bmatrix} Q_{m1} \\ Q_{m2} \end{bmatrix}_{local\ m} = \begin{bmatrix} \cos\theta_{mx} & \cos\theta_{my} & \cos\theta_{mz} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{mx} & \cos\theta_{my} & \cos\theta_{mz} \end{bmatrix} \cdot \begin{bmatrix} F_{m1} \\ F_{m2} \\ F_{m3} \\ F_{m4} \\ F_{m5} \\ F_{m6} \end{bmatrix}$$

$$\begin{bmatrix} u_{m1} \\ u_{m2} \end{bmatrix}_{local\,m} = \begin{bmatrix} \cos\theta_{mx} & \cos\theta_{my} & \cos\theta_{mz} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{mx} & \cos\theta_{my} & \cos\theta_{mz} \end{bmatrix} \cdot \begin{bmatrix} v_{m1} \\ v_{m2} \\ v_{m3} \\ v_{m4} \\ v_{m5} \\ v_{m6} \end{bmatrix}$$

Those two equations can be symbolically expressed as $\boldsymbol{Q_m} = \boldsymbol{T_m} \cdot \boldsymbol{F_m}$ and $\boldsymbol{u_m} = \boldsymbol{T_m} \cdot \boldsymbol{v_m}$ with the transformation matrix $\boldsymbol{T_m}$ given by:

$$\boldsymbol{T_m} = \begin{bmatrix} \cos\theta_{mx} & \cos\theta_{my} & \cos\theta_{mz} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{mx} & \cos\theta_{my} & \cos\theta_{mz} \end{bmatrix}$$

The transpose of the transformation matrix can be used to express the inverse equations between local and global coordinates systems: $\boldsymbol{F_m} = \boldsymbol{T_m}^{\boldsymbol{T}} \cdot \boldsymbol{Q_m}$ and $\boldsymbol{v_m} = \boldsymbol{T_m}^{\boldsymbol{T}} \cdot \boldsymbol{u_m}$ where $\boldsymbol{T_m}^{\boldsymbol{T}}$ is the transpose of the transformation matrix $\boldsymbol{T_m}$ expressed as follows:

$$\boldsymbol{T_m}^{\boldsymbol{T}} = \begin{bmatrix} \cos\theta_{mx} & 0 \\ \cos\theta_{my} & 0 \\ \cos\theta_{mz} & 0 \\ 0 & \cos\theta_{mx} \\ 0 & \cos\theta_{my} \\ 0 & \cos\theta_{mz} \end{bmatrix}$$

By using the member force-displacement relation in the local coordinate system and the transformation relations, the member force-displacement relation can be written in the global coordinate system; that is in function of $\boldsymbol{F_m}$ and $\boldsymbol{v_m}$. Using the equation $\boldsymbol{Q_m} = \boldsymbol{T_m} \cdot \boldsymbol{F_m}$ and $\boldsymbol{u_m} = \boldsymbol{T_m} \cdot \boldsymbol{v_m}$ and their inverses, the member stiffness relation $\boldsymbol{Q_m} = \boldsymbol{k_m} \cdot \boldsymbol{u_m}$ can be written as follows:

$$\boldsymbol{F_m} = \boldsymbol{T_m}^{\boldsymbol{T}} \cdot \boldsymbol{k_m} \cdot \boldsymbol{T_m} \cdot \boldsymbol{v_m}$$

This equation can be conveniently expressed as:

$$\boldsymbol{F_m} = \boldsymbol{K_m} \cdot \boldsymbol{v_m}$$

Where $\boldsymbol{K_m}$ is called the member stiffness matrix in the global coordinate and can be written as follows:

$$\boldsymbol{K_m} = \boldsymbol{T_m}^{\boldsymbol{T}} \cdot \boldsymbol{k_m} \cdot \boldsymbol{T_m}$$

$$\boldsymbol{K_m} = \frac{EA}{L}\begin{bmatrix} \cos^2\theta_{mx} & \cos\theta_{mx}\cdot\cos\theta_{my} & \cos\theta_{mx}\cdot\cos\theta_{mz} & -\cos^2\theta_{mx} & -\cos\theta_{mx}\cdot\cos\theta_{my} & -\cos\theta_{mx}\cdot\cos\theta_{mz} \\ \cos\theta_{my}\cos\theta_{mx} & \cos^2\theta_{my} & \cos\theta_{my}\cdot\cos\theta_{mz} & -\cos\theta_{my}\cdot\cos\theta_{mx} & -\cos^2\theta_{my} & -\cos\theta_{my}\cdot\cos\theta_{mz} \\ \cos\theta_{mz}\cdot\cos\theta_{mx} & \cos\theta_{mz}\cdot\cos\theta_{my} & \cos^2\theta_{mz} & -\cos\theta_{mz}\cdot\cos\theta_{mx} & -\cos\theta_{mz}\cdot\cos\theta_{my} & -\cos^2\theta_{mz} \\ -\cos^2\theta_{mx} & -\cos\theta_{mx}\cdot\cos\theta_{my} & -\cos\theta_{mx}\cdot\cos\theta_{mz} & \cos^2\theta_{mx} & \cos\theta_{mx}\cdot\cos\theta_{my} & \cos\theta_{mx}\cdot\cos\theta_{mz} \\ -\cos\theta_{my}\cdot\cos\theta_{mx} & -\cos^2\theta_{my} & -\cos\theta_{my}\cdot\cos\theta_{mz} & \cos\theta_{my}\cdot\cos\theta_{mx} & \cos^2\theta_{my} & \cos\theta_{my}\cdot\cos\theta_{mz} \\ -\cos\theta_{mz}\cdot\cos\theta_{mx} & -\cos\theta_{mz}\cdot\cos\theta_{my} & -\cos^2\theta_{mz} & \cos\theta_{mz}\cdot\cos\theta_{mx} & \cos\theta_{mz}\cdot\cos\theta_{my} & \cos^2\theta_{mz} \end{bmatrix}$$

3.2.5 Structure stiffness relations. The next step after having determined the member force-displacement relationship in the global coordinate system is to define the stiffness relation for the whole structure; that is to express the external load vector $\boldsymbol{P}$ as a function of the joint displacement vector $\boldsymbol{d}$.

Hereafter, it will be shown in three steps how to relate $\boldsymbol{P}$ and $\boldsymbol{d}$. For the purpose of clarity those steps will be applied to the truss shown in Figure 3.5.

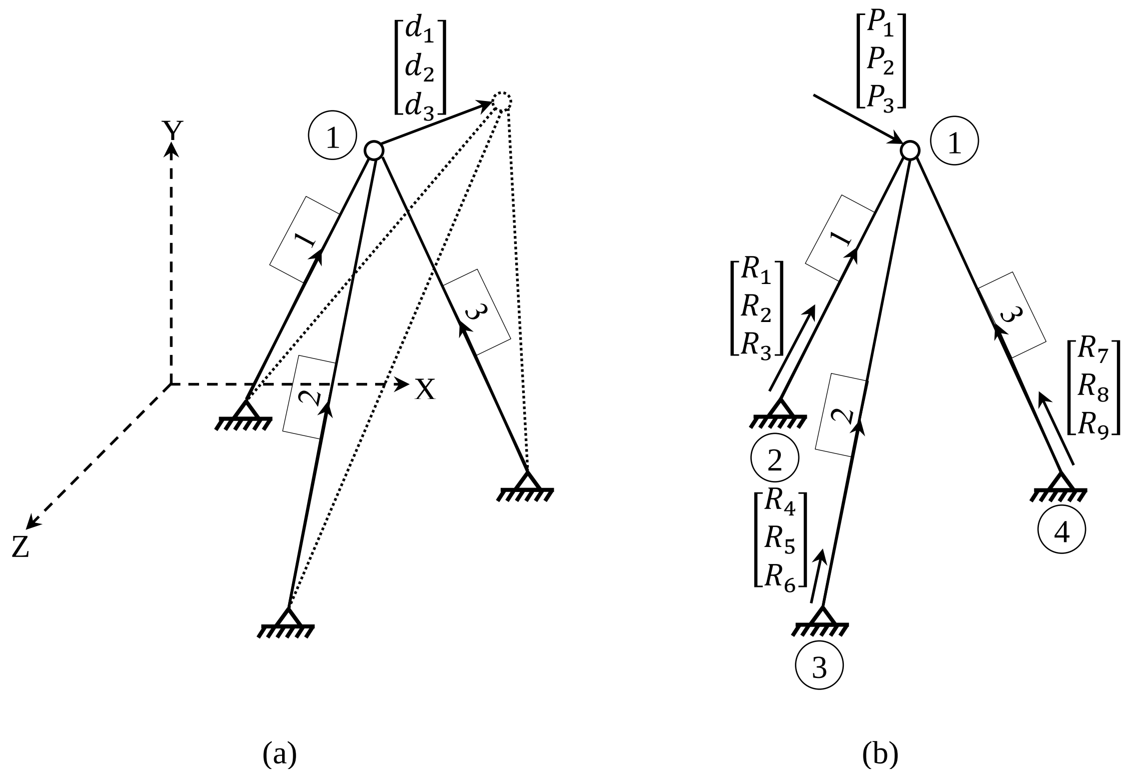


Figure 3.5 Displacement vector, load vector and reaction vectors

Step 1: The joint load vector $\boldsymbol{P}$ is expressed in terms of the member end forces vectors $\boldsymbol{F_m}$ in the global system by applying the equations of equilibrium for the joints of the structure.

$$\begin{cases} P_1 = F_{14} + F_{24} + F_{34} \\ P_2 = F_{15} + F_{25} + F_{35} \\ P_3 = F_{16} + F_{26} + F_{36} \end{cases}$$

Step 2: The joint displacement vector $\boldsymbol{d}$ is expressed in function of the member end displacements vector $\boldsymbol{v}$ in the global coordinate system by using the compatibility conditions, i.e. the displacements of the member ends must be the same as the corresponding joint displacements.

$$\begin{cases} v_{11} = 0 \\ v_{12} = 0 \\ v_{13} = 0 \end{cases} \quad \begin{cases} v_{21} = 0 \\ v_{22} = 0 \\ v_{23} = 0 \end{cases} \quad \begin{cases} v_{31} = 0 \\ v_{32} = 0 \\ v_{33} = 0 \end{cases} \quad \begin{cases} v_{14} = v_{24} = v_{34} = d_1 \\ v_{15} = v_{25} = v_{35} = d_2 \\ v_{16} = v_{26} = v_{36} = d_3 \end{cases}$$

Step 3: First, the compatibility equations obtained in step 2 are substituted into the member force-displacement relation in order to express the members' global end forces in function of the joint displacements. Then these forces are substituted into the joint equilibrium equations obtained in step 1 to establish the structure stiffness relationship between the joint loads vector $\boldsymbol{P}$ and the joint displacements vector $\boldsymbol{d}$. The member global stiffness relation for an arbitrary member m, $\boldsymbol{F_m} = \boldsymbol{K_m} \cdot \boldsymbol{v_m}$, can be written as:

$$\begin{bmatrix} F_{m1} \\ F_{m2} \\ F_{m3} \\ F_{m4} \\ F_{m5} \\ F_{m6} \end{bmatrix} = \begin{bmatrix} K_{m11} & K_{m12} & K_{m13} & K_{m14} & K_{m15} & K_{m16} \\ K_{m21} & K_{m22} & K_{m23} & K_{m24} & K_{m25} & K_{m26} \\ K_{m31} & K_{m32} & K_{m33} & K_{m34} & K_{m35} & K_{m36} \\ K_{m41} & K_{m42} & K_{m43} & K_{m44} & K_{m45} & K_{m46} \\ K_{m51} & K_{m52} & K_{m53} & K_{m54} & K_{m55} & K_{m56} \\ K_{m61} & K_{m62} & K_{m63} & K_{m64} & K_{m65} & K_{m66} \end{bmatrix} \cdot \begin{bmatrix} v_{m1} \\ v_{m2} \\ v_{m3} \\ v_{m4} \\ v_{m5} \\ v_{m6} \end{bmatrix}$$

And, using the relationships from step 2 to write the member end forces needed in the step 1:

$$\begin{cases} F_{14} = K_{144} \cdot v_{14} + K_{145} \cdot v_{15} + K_{146} \cdot v_{16} \\ F_{24} = K_{244} \cdot v_{24} + K_{245} \cdot v_{25} + K_{246} \cdot v_{26} \\ F_{34} = K_{344} \cdot v_{34} + K_{345} \cdot v_{35} + K_{346} \cdot v_{36} \\ F_{15} = K_{154} \cdot v_{14} + K_{155} \cdot v_{15} + K_{156} \cdot v_{16} \\ F_{25} = K_{254} \cdot v_{24} + K_{255} \cdot v_{25} + K_{256} \cdot v_{26} \\ F_{35} = K_{354} \cdot v_{34} + K_{355} \cdot v_{35} + K_{356} \cdot v_{36} \\ F_{16} = K_{164} \cdot v_{14} + K_{165} \cdot v_{15} + K_{166} \cdot v_{16} \\ F_{26} = K_{264} \cdot v_{24} + K_{265} \cdot v_{25} + K_{266} \cdot v_{26} \\ F_{36} = K_{364} \cdot v_{34} + K_{365} \cdot v_{35} + K_{366} \cdot v_{36} \end{cases}$$

$$\begin{cases} F_{14} = K_{144} \cdot d_1 + K_{145} \cdot d_2 + K_{146} \cdot d_3 \\ F_{24} = K_{244} \cdot d_1 + K_{245} \cdot d_2 + K_{246} \cdot d_3 \\ F_{34} = K_{344} \cdot d_1 + K_{345} \cdot d_2 + K_{346} \cdot d_3 \\ F_{15} = K_{154} \cdot d_1 + K_{155} \cdot d_2 + K_{156} \cdot d_3 \\ F_{25} = K_{254} \cdot d_1 + K_{255} \cdot d_2 + K_{256} \cdot d_3 \\ F_{35} = K_{354} \cdot d_1 + K_{355} \cdot d_2 + K_{356} \cdot d_3 \\ F_{16} = K_{164} \cdot d_1 + K_{165} \cdot d_2 + K_{166} \cdot d_3 \\ F_{26} = K_{264} \cdot d_1 + K_{265} \cdot d_2 + K_{266} \cdot d_3 \\ F_{36} = K_{364} \cdot d_1 + K_{365} \cdot d_2 + K_{366} \cdot d_3 \end{cases}$$

Finally using the relationships given in step 1:

$$\begin{cases} P_1 = F_{14} + F_{24} + F_{34} \\ P_2 = F_{15} + F_{25} + F_{35} \\ P_3 = F_{16} + F_{26} + F_{36} \end{cases}$$

$$\begin{cases} P_1 = (K_{144} + K_{244} + K_{344}) \cdot d_1 + (K_{145} + K_{245} + K_{345}) \cdot d_2 + (K_{146} + K_{246} + K_{346}) \cdot d_3 \\ P_2 = (K_{154} + K_{254} + K_{354}) \cdot d_1 + (K_{155} + K_{255} + K_{355}) \cdot d_2 + (K_{156} + K_{256} + K_{356}) \cdot d_3 \\ P_3 = (K_{164} + K_{264} + K_{364}) \cdot d_1 + (K_{165} + K_{265} + K_{365}) \cdot d_2 + (K_{166} + K_{266} + K_{366}) \cdot d_3 \end{cases}$$

These equations can be conveniently written in matrix from as:

$$\boldsymbol{P} = \boldsymbol{S} \cdot \boldsymbol{d}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} K_{144} + K_{244} + K_{344} & K_{145} + K_{245} + K_{345} & K_{146} + K_{246} + K_{346} \\ K_{154} + K_{254} + K_{354} & K_{155} + K_{255} + K_{355} & K_{156} + K_{256} + K_{356} \\ K_{164} + K_{264} + K_{364} & K_{165} + K_{265} + K_{365} & K_{166} + K_{266} + K_{366} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

The matrix $\boldsymbol{S}$ is called the structure stiffness matrix. The dimension of the square matrix $\boldsymbol{S}$ is equal to the number of degrees of freedom. $\boldsymbol{S}$ is a symmetric matrix. The foregoing equation between the degrees of freedom and the loading is called the structure stiffness relation and enables to calculate the structure's displacements for any given loading. Once the deformed shape of the structure is known, the member end forces can be calculated by using the member stiffness relations. Then these member end forces are written in their member local coordinate systems in order to find the value of the axial forces along the individual members.

The displacement method is a straightforward method for calculating the displacements and the forces inside a truss structure. The described procedure is easy to implement, since it only involves solving a matrix system with parameters properly defined from the structure's geometry.

The structure's stiffness coefficient of a joint in a direction equals the algebraic sum of the member stiffness coefficient, for member ends connected to the joint. This indicates that the structure's stiffness matrix can be formulated directly by adding the elements of the member stiffness matrices. This is made using numbering techniques which enable us to assemble the structure's stiffness matrix from the member stiffness matrices. The figure below provides an example of numbering techniques used to assemble the structure stiffness matrix.

$$\boldsymbol{K_1} = \begin{array}{cc} \begin{matrix} 2_x & 2_y & 2_z & 1_x & 1_y & 1_z \end{matrix} & \\ \begin{bmatrix} K_{111} & K_{112} & K_{113} & K_{114} & K_{115} & K_{116} \\ K_{121} & K_{122} & K_{123} & K_{124} & K_{125} & K_{126} \\ K_{131} & K_{132} & K_{133} & K_{134} & K_{135} & K_{136} \\ K_{141} & K_{142} & K_{143} & K_{144} & K_{145} & K_{146} \\ K_{151} & K_{152} & K_{153} & K_{154} & K_{155} & K_{156} \\ K_{161} & K_{162} & K_{163} & K_{164} & K_{165} & K_{166} \end{bmatrix} & \begin{matrix} 2_x \\ 2_y \\ 2_z \\ 1_x \\ 1_y \\ 1_z \end{matrix} \end{array}$$

$$\boldsymbol{K_2} = \begin{array}{cc} \begin{matrix} 2_x & 2_y & 2_z & 1_x & 1_y & 1_z \end{matrix} & \\ \begin{bmatrix} K_{211} & K_{212} & K_{213} & K_{214} & K_{215} & K_{216} \\ K_{221} & K_{222} & K_{223} & K_{224} & K_{225} & K_{226} \\ K_{231} & K_{232} & K_{233} & K_{234} & K_{235} & K_{236} \\ K_{241} & K_{242} & K_{243} & K_{244} & K_{245} & K_{246} \\ K_{251} & K_{252} & K_{253} & K_{254} & K_{255} & K_{256} \\ K_{261} & K_{262} & K_{263} & K_{264} & K_{265} & K_{266} \end{bmatrix} & \begin{matrix} 2_x \\ 2_y \\ 2_z \\ 1_x \\ 1_y \\ 1_z \end{matrix} \end{array}$$

$$\boldsymbol{K_3} = \begin{array}{cc} \begin{matrix} 2_x & 2_y & 2_z & 1_x & 1_y & 1_z \end{matrix} & \\ \begin{bmatrix} K_{311} & K_{312} & K_{313} & K_{314} & K_{315} & K_{316} \\ K_{321} & K_{322} & K_{323} & K_{324} & K_{325} & K_{326} \\ K_{331} & K_{332} & K_{333} & K_{334} & K_{335} & K_{336} \\ K_{341} & K_{342} & K_{343} & K_{344} & K_{345} & K_{346} \\ K_{351} & K_{352} & K_{353} & K_{354} & K_{355} & K_{356} \\ K_{361} & K_{362} & K_{363} & K_{364} & K_{365} & K_{366} \end{bmatrix} & \begin{matrix} 2_x \\ 2_y \\ 2_z \\ 1_x \\ 1_y \\ 1_z \end{matrix} \end{array}$$

$$\boldsymbol{S} = \begin{array}{cc} \begin{matrix} 1_x & 1_y & 1_z \end{matrix} & \\ \begin{bmatrix} K_{144} + K_{244} + K_{344} & K_{145} + K_{245} + K_{345} & K_{146} + K_{246} + K_{346} \\ K_{154} + K_{254} + K_{354} & K_{155} + K_{255} + K_{355} & K_{156} + K_{256} + K_{356} \\ K_{164} + K_{264} + K_{364} & K_{165} + K_{265} + K_{365} & K_{166} + K_{266} + K_{366} \end{bmatrix} & \begin{matrix} 1_x \\ 1_y \\ 1_z \end{matrix} \end{array}$$

Figure 3.6 Assembling of the structure stiffness matrix

3.3 Classical method to compute the probability of failure of a structure using failure paths

3.3.1 Framework. In the reliability analysis of a structural system, the probability of no structural failure under the applied loads is investigated. To do so, one should specify the probability of the failure of the structure's members and those of the failure paths.

The development of system failure paths requires structural analysis of the intact structure to determine the forces within the truss system. Given this force distribution, a probabilistic analysis using the loads and the member resistances as random variables is conducted to compute the probability of failure for every single member.

In practice, real structures can have numerous members; and thus it is not possible to enumerate and compute a probabilistic analysis for all of them. In these cases, some studies have shown that the structural reliability, i.e. the probability of no failure, can be estimated efficiently by using only dominant failure members. However, in this presented method, all the members are considered as candidate to failure; this leads to a complete enumeration of all the failure paths.

Then, to perform the next step which is to compute a member probabilistic analysis for the damaged structure, subsequent structural analyses are required. To do so, a post-failure behavior must be specified for the failed member. Usually, two different idealized behaviors are chosen to describe the constitutive law within the failed member. The first behavior is the brittle failure which considers that a failed component does not yield and cannot carry any additional load. In this case, the component is removed from the truss geometry in the structural analysis. The second failure is the ductile failure which in the case of an idealized perfectly plastic material considers that the member cannot carry any additional load but can accommodate infinite displacements. In this case, the component is removed from the truss geometry and replaced by a load equal to the force carried by the failed member corresponding to yielding. In this presented method, the fracture is conservatively and consistently set as a brittle failure.

Structural analysis and probabilistic analysis are alternately computed following the process described above until the formation of a failure mechanism in the structure, i.e. the structure has become unstable. Such a mechanism occurs when the structure's stiffness matrix becomes singular and has its determinant equals to zero. Thus the given failure path has reached to it end.

When all the failure paths have been found and the reliability analysis at the member level has been computed for all members in all scenarios, the structure's reliability is computed. Further discussions on the structure reliability evaluation from the member reliability will be provided later in this chapter.

To picture the variety of the failure paths, different diagrams can be used. One such diagram was described in chapter 2 (i.e., the failure path diagram). Another one is an analogy with series and parallel systems to depict the possible failures of the structure. In a statically indeterminate truss, the member failures are considered as parallel failures since only if all the member of a given failure path occur, the structure will fail. On the other hand, when considering the occurrence failure paths leading to the failure of the structure, failure paths are said to be in series.

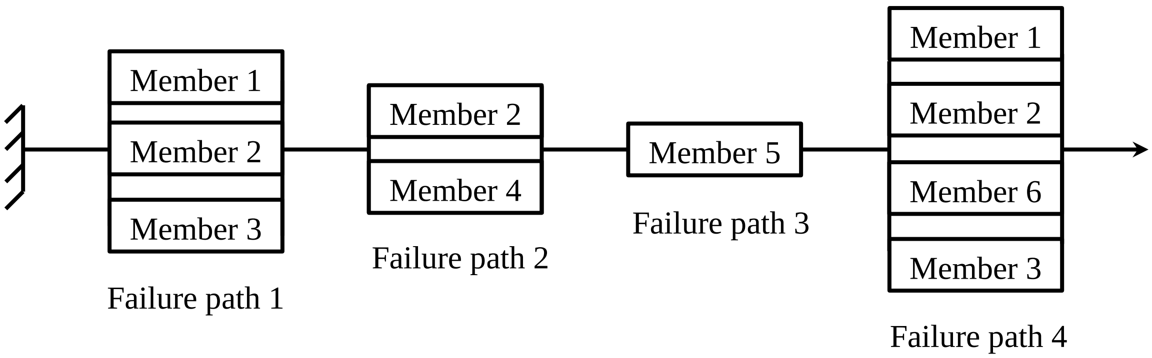


Figure 3.7 Parallel and series systems

3.3.2 Component reliability. In structural codes, design equations are written at the member level which is considered as the controlling element. To do so, effects of structural parameters such as loading, geometry and materials must be expressed at the member level; and structural theories must be developed to fulfill this purpose. Thus, most of the structural knowledge delivered by the codes is related to the member. The reliability analysis aims to be a global analysis of the structure; however, to benefit from most of the state of knowledge in structural engineering, the reliability analysis must comply with design requirements and be performed, at least for a primary approach, at the component level.

A structural component is subject to forces which, if exceeding a given level called member resistance, will cause its failure. Several modes of failure such as tensile failure, compressive failure, shear failure, stability failure, fatigue failure can occurs for the same member. These potential failure modes can be written as a limit state in terms of the member resistance R and the load effect on the considered member S. Each limit states must be considered in evaluating the probability of failure of the component. Thus, the probability of failure of a component i with n potential failure modes can be written as follows:

$$P_{fc_i} = P\left(M_{i_1} \cup M_{i_2} \cup \dots \cup M_{i_n}\right)$$

Where:

- M_{i_m} represents the event of failure for the component i in its mode of failure m.

Because of the correlation between modes of failure the component probability of failure cannot be calculated using this formulation. Given some mathematical conditions concerning the events of failure, it can be shown that the component probability of failure is bounded by two extreme situations. If all the failure modes are perfectly correlated:

$$P_{fc_i} = \max(P\left(M_{i_1}\right), P\left(M_{i_2}\right), \dots, P\left(M_{i_n}\right))$$

And if all the failure modes are statistically independent:

$$P_{fc_i} = 1 - \prod_{i=1}^{n}(1 - P\left(M_{i_i}\right))$$

These two cases could be chosen as two bounds for the actual P_{fc_i}. Indeed, in reality the failure modes are neither perfectly correlated nor statistically independent. However, in the case of the failure of a single member, the failure modes written in terms of limit states are highly correlated because the same random are used to write the resistance and the load effect for the different modes of failure. Thus, the case of perfectly correlated modes of failure can be taken as an acceptable approximation of the member probability of failure. In this study, an additional approximation is considered. This is in assuming that the controlling mode of failure is either the tensile or compressive mode. A member is said to have failed when it has started yielding. This approximation is made to simplify the analysis and is exact in most cases. The yielding mode of failure can be written in term of R and S as follows:

$$R < S$$

Where R is the variable representing the member resistance whereas S is the variable corresponding to the load effects on the considered member. The variables R and S are defined by using a set of variables representing the loading and the structure geometry and materials. The component is considered to have yielded when R is less than S. The equality between the two variables defines the limit state. A performance function Z can be introduced as:

$$Z = R - S$$

The limit state is defined by the zeros of the performance function and the member is considered to be failed when Z is negative. Assuming statistical independence between R and S, which is close to the reality given that R and S are not written with the same variables and considering that these basic random are statistically independent, the following properties can be established for the performance function Z.

$$\mathbb{E}(Z) = \mathbb{E}(R) - \mathbb{E}(S)$$

$$\sigma(Z) = \sqrt{\sigma(R)^2 + \sigma(S)^2}$$

Where:

- $\mathbb{E}(Z), \mathbb{E}(R)$ and $\mathbb{E}(S)$ are respectively the expected value the performance function, the resistance and the load effect the considered member.
- $\sigma(Z)$, $\sigma(R)$ and $\sigma(S)$ are respectively the standard deviation of the performance function, the resistance and the load effect on the considered member.

Further calculations require assigning a distribution function to the performance function variable. The normal distribution function is chosen as a realistic and simple candidate.

$$Z \sim \mathcal{N}(\mathbb{E}(Z), \sigma(Z)^2)$$

Using the linear transformation below, it can be shown that:

$$\frac{Z - \mathbb{E}(Z)}{\sigma(Z)} \sim \mathcal{N}(0,1)$$

Where $\mathcal{N}$ is the symbol used to refer to a normal distribution function. The parameters of the normal distribution of Z are the expected value and the variance $\sigma(Z)^2$. $\mathcal{N}(0,1)$ is the standard normal distribution, i.e. a normal distribution with zero as an expected value and a variance equals to one.

Using the performance function to represent the failure, the failure probability is given as follow:

$$P_{fc_i} = P(Z_i \leq 0) = \Phi_{(\mathbb{E}(Z), \sigma(Z)^2)}(0)$$

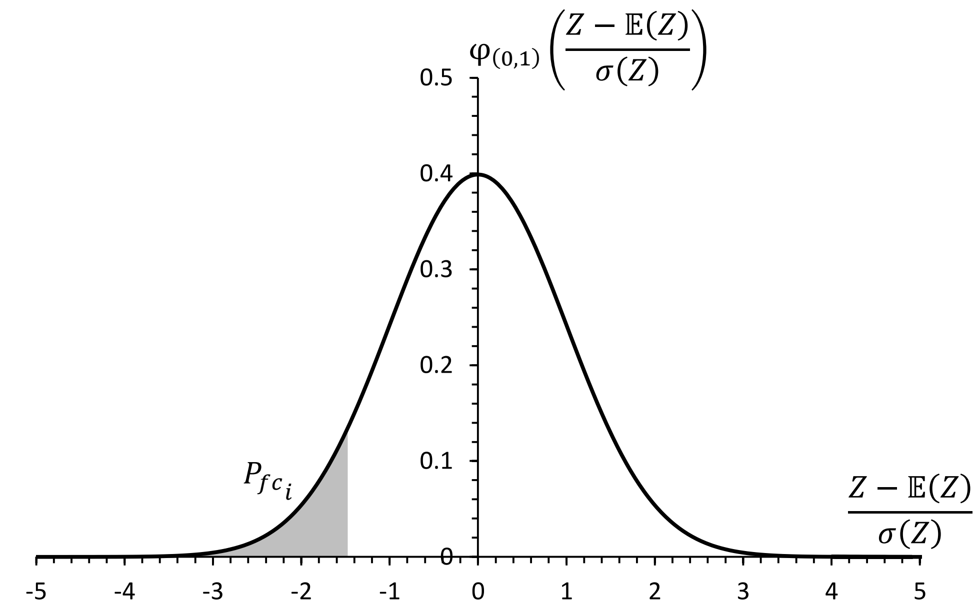
Where $\Phi_{(\mathbb{E}(Z),\sigma(Z)^2)}$ is the cumulative distribution function of the normal distribution function assigned to the performance function Z. Using the linear transformation showed above:

$$P_{fc_i} = \Phi_{(0,1)}\left(\frac{0 - \mathbb{E}(Z)}{\sigma(Z)}\right) = \Phi_{(0,1)}\left(\frac{-(\mathbb{E}(R) - \mathbb{E}(S))}{\sqrt{\sigma(R)^2 + \sigma(S)^2}}\right)$$

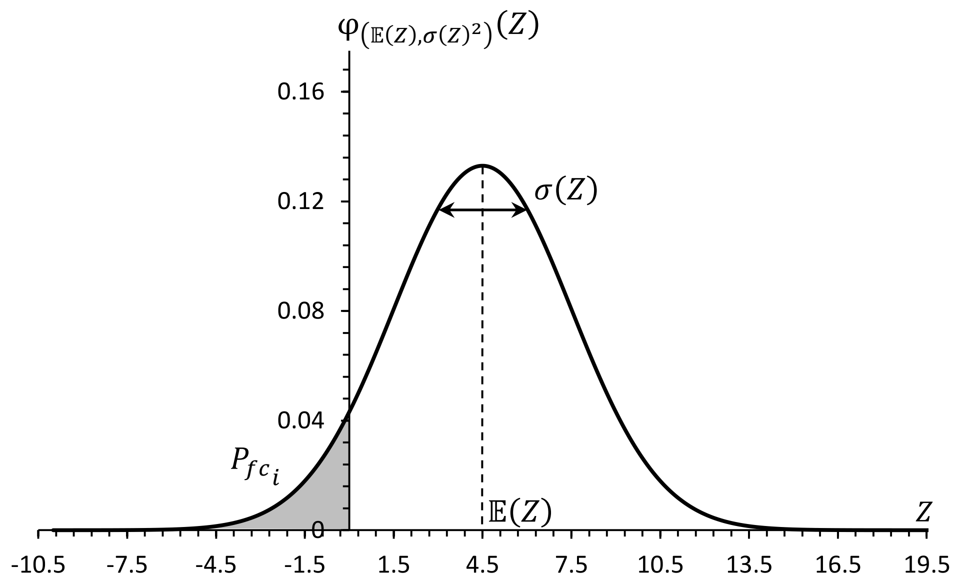
The probability failure of the member i is defined in terms of its resistance and load effect expected value and standard deviation.

The inverse of the probability of failure is the probability of survival which is the reliability of the member and is written as follows:

$$P_{sc_i} = 1 - P_{fc_i}$$



(a) Standard normal distribution



(b) Normal distribution of Z

Figure 3.8 Member probability of failure associated with Z

3.3.3 Structure's reliability. In an indeterminate structure, the failure of a component does not necessarily leads to the failure of the structure. In fact, the failure is cause by a succession of failures among structural members, which is called a failure path. Numerous failure paths can exist within a structure. To compute the probability of failure of the structure, one should be able, first, to quantify the probability of occurrence of all single failure paths and, then, to prescribe a method to evaluate the probability of failure of the structure from those of its failure paths.

Owing to the above proposed framework, the reliability of every member in all possible scenarios can be computed. These member-related data are the only available data to compute the structure's reliability.

First, the probability of occurrence of a failure path must be computed. This requires the calculation of the probability of multiple events using the conditional probability. The probability of the occurrence of the failure mechanism i (starting by the failure of the member 1 and ending with the failure of member n, can be written as follows:

$$P_{fm_i} = P_{fc_1} \times P_{fc_{2|C_1}} \times P_{fc_{3|C_1,C_2}} \times \dots \times P_{fc_{n|C_1,C_2,\dots C_{n-1}}}$$

Where P_{fc_1} is the probability of failure of the first component in the failure path for the intact structure, $P_{fc_{2|C_1}}$ is the probability of failure of the second component given that the first component has failed and $P_{fc_{n|C_1,C_2,\dots C_{n-1}}}$ is the probability of failure of the last component of the failure path given that the other component has failed first. These probabilities are evaluated using the member reliability for all scenarios considered in the structural analysis framework.

Then the probability of failure of the structure is evaluated from those of its failure paths. A structural system with k failure paths will have a probability of failure written as:

$$Pf = P(m_1 \cup m_2 \cup \ldots \cup m_k)$$

Where m_i correspond to the occurrence of the failure path denoted by i. As it was shown for the different modes of failure of a single member, the evaluation of the probability of failure of the structure is impractical because of the correlation between failure paths. It was also shown that such a probability can be bounded as follows:

$$\max(P(m_1), P(m_2), \ldots, P(m_k) \leq Pf \leq 1 - \prod_{i=1}^{k}(1 - P(m_i))$$

In some studies, affirming that the two bounds are close, it was permitted to approximate Pf with the two bounds average:

$$Pf = \frac{\max(P(m_1), P(m_2), \ldots, P(m_k) + 1 - \prod_{i=1}^{k}(1 - P(m_i))}{2}$$

The probability of survival of the structure is written in function of the probability of failure as follows:

$$Ps = 1 - Pf$$

3.3.4 Shortcomings of the proposed method. The main objective of this chapter was to provide the reader a comprehensive approach to a classical method to compute the probability of failure of a structure. Thus, the reader was made aware of the assumptions made, the reliability and structural theories used were introduced and the global framework leading to the computation of the structure probability of failure was explained. This method is said to be classical because most of the introduced hypotheses are those used in the structural reliability literature. However, this method, besides providing a comprehensive approach, has several shortcomings as explained below.

3.3.4.1 The method is time consuming. In case of a real structure, due to the number of members, numerous failure paths are to be computed. At every stage of every failure path a structural analysis and a reliability analysis must be done. In practice the time needed by a computer to provide the structure's probability of failure following this method increases drastically with the number of components and become impractical for real structure. It was demonstrated in [13] that a good approximation of the probability failure can be given using automatic generated stochastically dominant failure paths. This results in less effort in conducting structural and reliability analysis. It was also shown that the structural analyses can be done more efficiently using the force method instead of the displacement method [7]. In [7] the authors present an improved force method which is easy to implement and reduces the number of calculations to conduct a structural analysis. Among all those efforts to reduce computational time, most of the methods take advantage of the fact that identical structural and reliability analyses are needed to be considered. Every stage of a failure path corresponds to a different scenario; however, if the order of member failure is ignored, scenarios were the same members are involved in the failure path can be considered as being identical. This can be done without influencing the final probability of failure because the structural and reliability analyses do not take into account the order in which members have failed. Thus ignoring the sequence of component failure can lead to faster computations.

3.3.4.2 Significance of correlation among member failures and paths. The classical methods are incapable of dealing with the correlation in member failure and in failure paths. Concerning the member failure, it is assumed individual member conditions are independent; and as such, the reliability analysis can be conducted for members as stand-alone structural components. However, it is obvious that because of the common loads and the fact that materials are often obtained from the same source, conditions among members may be partially correlated. When a reliability analysis is conducted on a single member, these load effect and properties correlations have no direct effect in the results, since the conditions among members are considered independent. However, looking globally at the structure, the event of failure of a member shares many common parameters with other members. This indicates that at least in some members, once one failed, they others have also failed. For example, consider two members sharing the same properties. Let's further assume that for every given load, member 1 is taking more load effect that member 2. Thus member 1 is more vulnerable to failure; and its failure probability will be evaluated according to its internal load and its resistance. However, member 2 will never fail in this particular scenario, since member 1 will need to fail first. The probability of failure of the member 2 is therefore zero, if member 1 has not failed yet. It is obvious that in reality this cannot be possible.

In the case of failure paths, it was recognized that they may be correlated. However, for simplicity, again, all failure paths were assumed to be independent. The two bounds thus obtained for the system failure probability correspond to the two extreme situations where either all the failure paths are totally correlated or they are totally independent. Madsen and Ditlevsen [2] elaborate further on this shortcoming and offer solutions to have more accurate bounds for the system failure probability. Moreover, it should be noticed that in the case of totally correlated failure paths, the corresponding bound is inconsistent with actual conditions, since in reality the paths are almost never perfectly correlated.

3.3.4.3 The load issue. In classical methods, in any scenarios, when the reliability analyses are computed in terms of the load effect S and the member resistance R, the characteristics of the random variables needed to compute S and R remain the same throughout the loading regime. These characteristics will need to be updated as the loading regime continues and the nature and number of failure mechanisms keep on changing. Using classical methods, a member failure will affect the structure geometry but not the random variables representing the member properties and the loading. This is an over-simplification. To clarify this shortcoming, consider the following example.

Two successive member failures are considered. Due to an unexpectedly high load, member 1 fails. Using the classical methods, one uses the estimates of the load and resistance and computes the probability of failure for member 1.. Now assume that this unexpectedly high load no longer applies on the structure. A redistribution of the loads occurs within the structure and thus a second reliability analysis can be conducted to evaluate the probability of failure of member 2. We realize that using this process, the unexpected high load (which affected member 1) will not damage the structure beyond member 1. However, the method will treat the system as though the unexpected high load is persistently applying on the structure.

In a realistic situation, a failure happens because of the simultaneous occurrences of an unexpectedly high load and weak member resistance. If the load causes the failure of a member (the first member failed) and starts a failure mechanism, there are chances that it will damage consecutively several other members. When reliability analyses are conducted separately, as it is the case in classical methods, the random variables governing the loading must be updated every time a new reliability analysis is conducted on the structure. In the same manner, because member properties are partially correlated among each other, the knowledge of upon a member failure, the random variable describing the properties of surviving members will need to be updated.

Among assumptions and simplifications made in classical methods, the one on constant load is perhaps affecting the final results for the probability of failure most. This is because; this assumption minimizes the effect of a strong load that may only unexpectedly apply on the structure, and missed when conducted the reliability analysis.

3.3.4.4 The issue related to the member post failure behavior. In this method, a failed member is a member which has started yielding. Once a member is said to be failed, it is considered as absent in the structure. Considering that a yielded member cannot carry any load is far from being true. In reality, when a member is yielding it can still at least carry the same load. In that model, the material redundancy is not considered at all. As it was mentioned earlier, the material redundancy corresponds to the additional load and additional displacement that a member can withstand after having yield. The material redundancy which corresponds to additional load is called the constraint redundancy and the redundancy which corresponds to additional displacement is called the displacement redundancy.

Experimentations are conducted on materials to plot the stress-strain constitutive relation of the considered material. For common materials, the shape of the stress-strain curve is well-know and mathematical model are formulated in order to use the material in structural analysis. Two simple models which take into account the material redundancy consider the material as:

- Linear elastic – perfectly plastic. An example of a stress-strain curve of such a material behavior is given by the figure 2.5 (b).
- Linear elastic – linear work hardening plastic. An example of a stress-strain curve of such a material behavior is given by the figure 2.5 (d).

Contrary to the classical method, these two descriptions of the material behavior are a good approximation of the real material behavior. In the research literature, numerous examples of reliability methods using the linear elastic – perfectly plastic approximation can be found. Indeed, this approximated behavior can be easily implemented in the structural analysis. When a member is considered as failed, instead of just removing it from the structure geometry as it is done in the classical method, two equal forces corresponding to the yielding strength of the member are added to the structural model as additional loads. These additional loads are located at the two pins where the failed member was attached before and act in the failed member direction.

3.3.4.5 The issue related to the member mode of failure. In this method, the mode of failure is selected. Probabilities are not computed at a member level in order to know which mode failure is dominant and what is the final probability of failure of the member given that all possible modes of failure are considered. Considering multiple mode of failure in a member would lead, because of the partial correlation between modes of failure, to the need of the computation of bounds.

3.3.5 Concluding remarks. Globally, the features to improve in this classical method of computing the probability of failure of a structure can be gathered in three main aspects which are

- The computational time of the method. Failure paths are numerous and analyses required are time consuming. The main issues regarding with this aspect are the number of scenarios to be considered and the computer time needed to compute all the structural analyses.
- The realistic representation of the failure mechanism. The framework in which analyses are conducted must compel to the reality of the structure failure mechanism. The main issues regarding with this aspect are the misrepresentation of the loading, the correlation between member failure and the post failure behavior of a member.
- The approximation made within the method. They must be precise and use as many information as possible. The main issue regarding this aspect is the bounds given for correlated failure path.

CHAPTER 4

MODIFIED METHOD FOR COMPUTING THE PROBABILITY OF FAILURE OF A STRUCTURE

4.1 Introductory remarks

The purpose of this chapter is to propose a modified method to compute the probability of failure of a structure based on the classical framework introduced in chapter 3. The method is based on revised hypotheses and new strategies to compute the reliability and structural analyses are introduced. The main objective is to provide an improved method with regards to the shortcomings of the classical methods. However, for the sake of simplicity, these modifications do not affect the classical method framework. This improved method uses materials taken from Ditlevsen and Madsen [2].

4.2 The limit state model

4.2.1 Limit state. A failure limit state is a load requirement beyond which a structure is said to be failed. This requirement is formulated with a mathematical model which usually includes, as input variables, the structure's geometry and mechanical properties as well as the applied loads. A scalar function is used to define a limit state as follow:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \mid f(\boldsymbol{x}) = a\,, a \in \mathbb{R}\}$$

Where,

- f is a scalar function defining the limit state
- $\boldsymbol{x}$ is the vector describing the structure and its load
- $\mathfrak{L}$ is a subset of $\mathbb{R}^n$

The components of the vector $\boldsymbol{x}$ are the independent variables that describe the structure's geometry, the structure mechanical properties and the applied loads. These variables are considered free in the sense that their values can be chosen freely and independently in the subset $\mathfrak{L}$ of $\mathbb{R}^n$. The subset $\mathfrak{L}$ is the domain of definition of the model. Every choice of $\boldsymbol{x}$ in $\mathfrak{L}$ corresponds to a uniquely-defined structure and its applied loads. Thus, $\boldsymbol{x}$ is a pure mathematical expression, which may not be physically possible. For example, when the loads exceed the capacity of the structure, the failure occurs; so physically the notion of load being larger than the resistance is no longer valid. The limit state defines two sets within, which are defined as follows:

- Safe set:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n | \, f(\boldsymbol{x}) > a \, , a \in \mathbb{R}\}$$

  In the safe set, the requirement defined by the limit state is said to be satisfied.

- Failure set:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \, | \, f(x) \leq a \, , a \in \mathbb{R}\}$$

  In the safe set, the requirement defined by the limit state is not satisfied.

The boundary of the safe set, which is the boundary of the boundary of the failure set also, is the limit state.

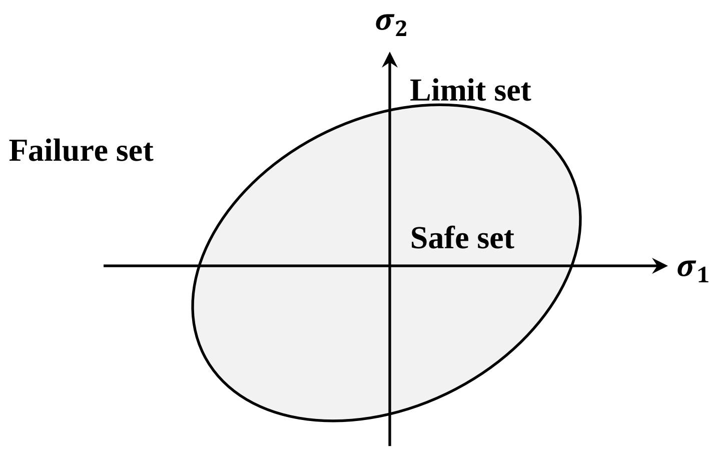


Figure 4.1 Von-Mises Criterion limit state

Figure 4.1 depicts the Von-Mises limit state, which is defined via the stress components σ_1 and σ_2, as well as the yield stress σ_y. The safe set and the failure set are a partition of $\mathfrak{L}$. This property is a consequence of the definition of f on $\mathfrak{L}$. Thereafter, for the sake of simplicity; the limit state will be represented by the zeros of the scalar function, that is $a = 0$. For later use, the scalar function f will be required to be piecewise differentiable on $\mathfrak{L}$. Moreover, the limit state scalar function will take positive values in the internal of the safe set and negative values in the internal of the failure set. It should be noted that the choice of f is not unique: the same limit state can be defined with a different scalar function.

Mainly, limit states can be of two different categories which are the failure limit state and the serviceability limit states. The first category represents a situation where the structure is about to lose its integrity. The structure will be at least damaged and the initial situation cannot be recovered. The second category corresponds to the limit situation where the structure is about to not be able to fulfill the requirement for which it was designed. This second situation is necessary in design; but since it is reversible it does not raise any reliability issues. In the hereafter, only failure limit state will be considered.

4.2.2 Linear limit state. A linear limit state is defined by a linear scalar function f, (also called linear form), as follows:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \mid f(\boldsymbol{x}) = 0\}$$

Where,

- f is the linear form defining the limit state. f is written as follows:

$$f(\boldsymbol{x}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + b$$

  - $a_1, a_2, \ldots, a_n, b$ are given constant coming from a structural analysis and a definition of the limit state.
  - $x_1, x_2, \ldots, x_n$ are the variables of the model.

The limit state defined an n-1 dimensional hyperplane which separates the domain of definition of the model in two sets: the safe set and the failure set.

Remark: The safe set is a convex set, which means that if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ belong to the safe set, for every t in the interval $[0; 1]$, $(1 - t)\boldsymbol{x}_1 + t\boldsymbol{x}_2$ is in the safe set. Owing to this property, convex sets, defined by a form linear or not, are particularly convenient for limit state analysis [2]. Moreover, any limit state having a convex safe set can be described as the intersection, finite or infinite, of linear limit states. Finally, it should be noted that in the case of linear limit, even the failure set is convex. The case of non-convex failure set is demonstrated in Figs. 4.3 and 4.4.

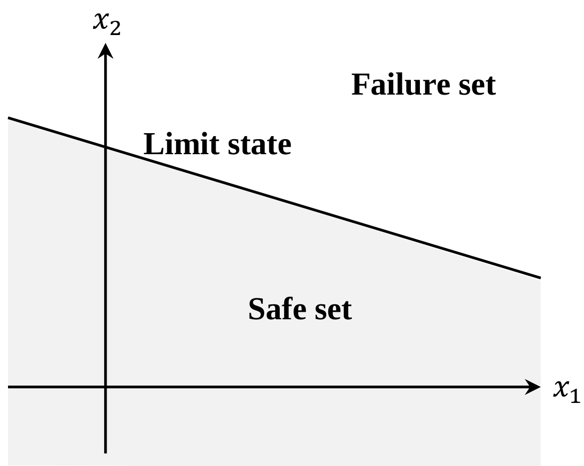


Figure 4.2 Linear limit state

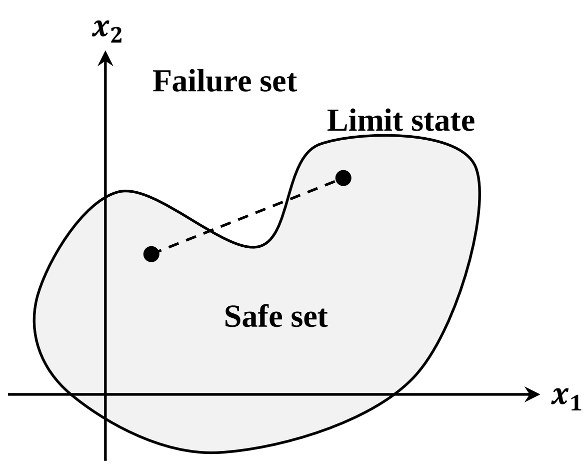


Figure 4.3 Non-convex safe set

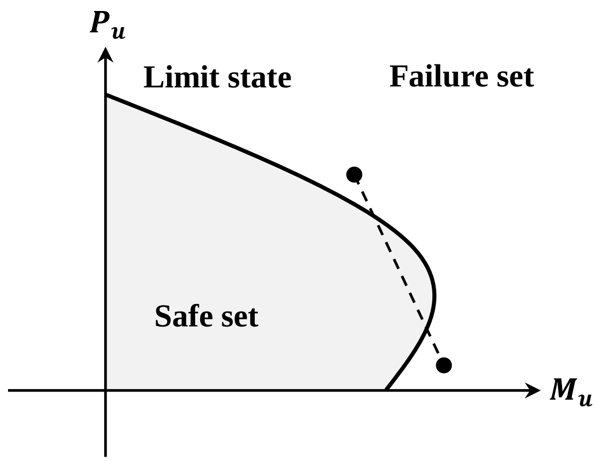


Figure 4.4 Non-convex failure set for a concrete column.

Hereafter the possibility to describe a convex set as an intersection of linear limit state will be used to approximate a non-liner limit state by a given number of linear limit states.

4.2.3 Multiple linear limit states. In practice, a structure has multiple modes of failure, which must be taken into account by the definition of multiple failure limit states (figure 4.5 and 4.6). These modes of failure stem, for instance, from the failure of different elements within the structure or from the different possible failure modes in a member, such as shear or bending moment modes of failure. Considering these modes of failure will lead to a larger domain of definition of the structure and load variables, since more variables will be needed to describe the model. In this domain, every mode of failure has a corresponding limit state function, which is a linear form in the case of linear limit state. The new safe set is the intersection of all the safe set corresponding to different single modes of failure.

For limit state i with regard to the mode of failure i:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \mid f_i(\boldsymbol{x}) = 0\}$$

In which,

- f_i is the linear forms corresponding to the limit state i
- $f_i(\boldsymbol{x}) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + b_i$

Safe set i when the limit state i is taken as a single limit state:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \mid f_i(\boldsymbol{x}) > 0\}$$

Safe set for a multiple linear limit state:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \mid \forall\, i, f_i(\boldsymbol{x}) > 0\} = \bigcap_i \{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \mid f_i(\boldsymbol{x}) > 0\}$$

The safe set for a multiple linear state is defined as the intersection of every single safe set.

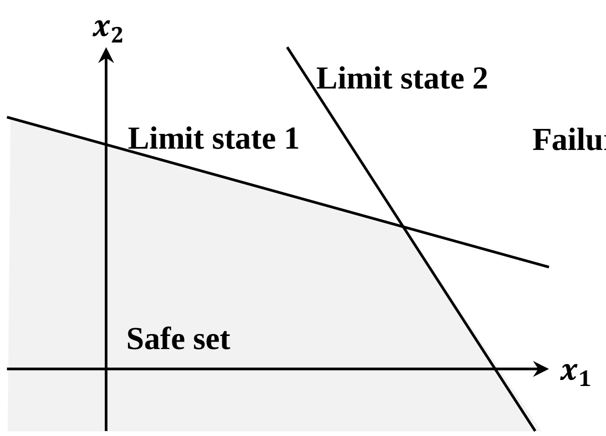


Figure 4.5 Multiple linear limit state in 2-dimensional space

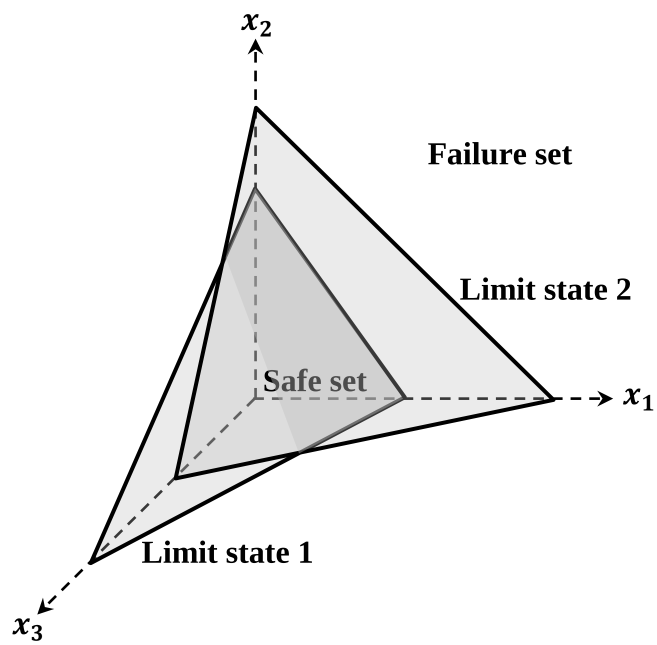


Figure 4.6 Multiple linear limit states in 3-dimensional space

4.3 Probability calculations for linear limit state

The limit state model described above is enough to design safe structures for deterministic variables, because in this model the variables describing the structure and its loads are either in the safe set or in the failure set. However, because of the randomness in design parameters, a model incorporating random variables is needed in order to measure the structure safety. In the probabilistic model, introduced next, the probability of $\boldsymbol{x}$ to be on the safe set will be defined as a measure of the safety or reliability of the structure. The part 4.3 of this study is taken from [2].

4.3.1 The second moment representation. Randomness of the input variable $\boldsymbol{x}$ is taken into account by modeling the coordinates of $\boldsymbol{x}$ as random variables with a mean and covariances as follows:

$$\boldsymbol{x} \rightarrow \boldsymbol{E}(\boldsymbol{x}) = \begin{pmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_n) \end{pmatrix} = \begin{pmatrix} E(x_1) & E(x_2) & \cdots & E(x_2) \end{pmatrix}^T$$

$$\boldsymbol{x} \rightarrow \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T}) = \boldsymbol{E}(\boldsymbol{x} \cdot \boldsymbol{x^T}) - \boldsymbol{E}(\boldsymbol{x}) \cdot \boldsymbol{E}(\boldsymbol{x})^{\boldsymbol{T}}$$

These two quantities are the second moment representation of the random vector $\boldsymbol{x}$. Using such a distribution has two main advantages. Firstly, because a minimal quantity of data is carried through the calculations, it leads to the most effective methods of computing the probabilities. Secondly, in practice, due to uncertainty, it is out of question to choose distribution types on a basis of solid data information. On the other hand, if one should choose a distribution, the second moment distribution can be used a first basis for a more precise description.

4.3.2 Limit state, safety margin and reliability index in the normalized space. The random approach of the limit state motivates the introduction of a new variable M which is the outcome of the linear form f.

$$M = \boldsymbol{a^T} \cdot \boldsymbol{x} + \boldsymbol{b}$$

$$M = a_1 \cdot x_1 + a_2 \cdot x_2 + \cdots + a_n \cdot x_n + b$$

The random variable M is called linear safety margin and a positive outcome of M indicates that $\boldsymbol{x}$ is in the safe set. However, the outcomes of M are arbitrary and do not represent the safety of the structure. Indeed, although the mean value of M indicates the mean distance from M to the limit state in $\mathbb{R}$, this distance has nothing to do with probabilities and does not enable to compare different limit states. To illustrate the fact that the outcomes of M are arbitrary, it should be recalled that the same limit state can be written in terms of different linear forms. In the case of linear forms, these different linear forms can be found after multiplication of a first linear form by an arbitrary constant. Thus, one of the purposes of this chapter is the establishment of an invariant number capable of comparing safety margin with regards to overpassing the limit state.

The objective is to interpret the mean distance from M to the limit state in $\mathbb{R}$, i.e. $E(M)$, as a consistent measure of safety. To do so, the standard deviation of M is taken as a unit, and the reliability index is introduced as follows:

$$\beta = \frac{E(M)}{D(M)} = \frac{\boldsymbol{a^T} \cdot \boldsymbol{E}(\boldsymbol{x}) + b}{\sqrt{\boldsymbol{a^T} \cdot \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T}) \cdot \boldsymbol{a}}}$$

This number is unchanged after a multiplication of the linear form defining the linear limit state. That means that all linear forms representing the same limit state are assigned with the same reliability index. Moreover, it can be shown that the number β is invariant under any regular inhomogeneous linear mapping of the random $\boldsymbol{x}$ into another random vector $\boldsymbol{y}$. [2]

$$\boldsymbol{x} = \boldsymbol{T} \cdot \boldsymbol{y} + \boldsymbol{c}$$

$$M = \boldsymbol{a^T} \cdot \boldsymbol{x} + \boldsymbol{b} = \boldsymbol{a^T} \cdot (\boldsymbol{T} \cdot \boldsymbol{y} + \boldsymbol{c}) + \boldsymbol{b} = \boldsymbol{a^T} \cdot \boldsymbol{T} \cdot \boldsymbol{y} + \boldsymbol{b} + \boldsymbol{a^T} \cdot \boldsymbol{c} = \boldsymbol{a'^T} \cdot \boldsymbol{Y} + \boldsymbol{b'}$$

Under a specific $\mathbb{R}^n$ linear mapping of the vector $\boldsymbol{x}$ a new vector $\boldsymbol{y}$ can be defined with the following properties:

- $\boldsymbol{E(y) = 0}$
- $\boldsymbol{Cov(y, y^T) = I}$

In this particular mapping space β has the following convenient expression:

$$\beta = \frac{E(M)}{D(M)} = \frac{\boldsymbol{a}'^{\boldsymbol{T}} \cdot \boldsymbol{E}(\boldsymbol{y}) + b'}{\sqrt{\boldsymbol{a}'^{\boldsymbol{T}} \cdot \boldsymbol{Cov}(\boldsymbol{y}, \boldsymbol{y}^{\boldsymbol{T}}) \cdot \boldsymbol{a}'}} = \frac{b'}{\sqrt{\boldsymbol{a}'^{\boldsymbol{T}} \cdot \boldsymbol{a}'}}$$

This final expression of the reliability index is convenient because it is also the distance from the origin to the hyperplane defined by the linear limit state in the specific mapped space just defined above. Hereafter, this space will be called the normalized space (figure 4.7).

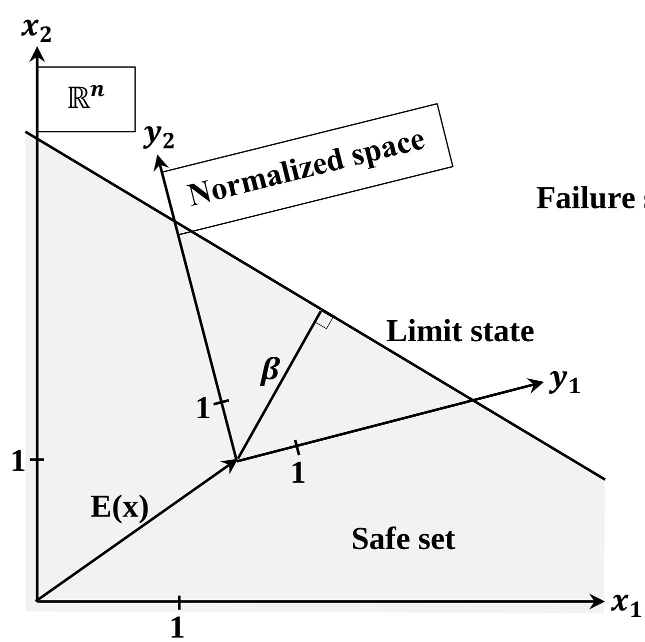


Figure 4.7 Geometric interpretation of the reliability index in the normalized space

The invariant number β can be interpreted as a measure of safety with respect to overpassing the limit state. According to its definition it measures the distance from the limit state represented by $M = 0$ to the mean value $E(M)$ with the standard deviation $D(M)$ taken as a unit. The parameter β enables to have an insight regarding safety of the structure with respect to the reliability; however it does not give a measure of the probability of overpassing the limit state. To do so, additional information concerning the input random variables, such as distribution functions must be provided.

4.3.3 Distribution function. The model described above includes a limit state and a random variable described by its second moment and defines a comparison criterion among limit states by using the reliability index. This criterion can be considered as a reasonable engineering reliability comparison, since it is enough to define a structure's safety condition with respects to reliability. However it is impossible to compute the probability of failure using only the safety index.

To compute probabilities, distribution functions must be assigned to the initial input random variables. For simplicity, the input random variables x_i are assumed to follow a normal density function. Thus, the random vector $\boldsymbol{x}$ follows a multivariate normal distribution of dimension n with the n-dimensional mean vector and the $n \cdot n$ covariance matrix.

$$\boldsymbol{x} \sim \mathcal{N}(\,\boldsymbol{E}(\boldsymbol{x})\,, \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T})\,)$$

$$p\Big(\boldsymbol{x};\ \boldsymbol{E}(\boldsymbol{x}), \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T})\Big) = \frac{1}{(2\pi)^{\frac{n}{2}} \cdot \sqrt{\boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T})}} \cdot e^{\left(-\frac{1}{2}\cdot\left(\boldsymbol{x}-\boldsymbol{E}(\boldsymbol{x})\right)^{T}\cdot \boldsymbol{Cov}(\boldsymbol{x},\boldsymbol{x^T})^{-\mathbf{1}}\cdot\left(\boldsymbol{x}-\boldsymbol{E}(\boldsymbol{x})\right)\right)}$$

This multivariate normal density function can be transformed into a standard normal multivariate density function. This conveniently enables us to conclude that $\boldsymbol{y}$ follows a standard normal multivariate density function.

$$\boldsymbol{x} \sim \mathcal{N}(\,\boldsymbol{E}(\boldsymbol{x})\,, \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{T}})\,) \Leftrightarrow \boldsymbol{y} \sim \mathcal{N}(\,\boldsymbol{0}\,, \boldsymbol{I}\,)$$

$$p(\boldsymbol{y};\ \boldsymbol{0}, \boldsymbol{I}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{\left(-\frac{1}{2}\cdot \boldsymbol{y}^{\boldsymbol{T}}\cdot \boldsymbol{y}\right)} = \varphi_n(\boldsymbol{y}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\left({y_1}^2+{y_2}^2+..+{y_n}^2\right)}$$

In the normalized space, the components of $\boldsymbol{y}$ are independent and have a variance equals to 1 with an expected value of 0.

The density of $\boldsymbol{y}$ is constant on every hypersphere centered at the origin of the normalized space. This is consistent with the reliability index which depends only on the distance between the considered hyperplane and the origin. This property which is called the rotation symmetry reflects a reasonable engineering comparison between limit states and is consistent with the second moment representation [2].

The distribution function for a given distance between the origin of the normalized space and the end of the vector $\boldsymbol{y}$ is calculated as follows:

$$p(\|y\| = r) = \oint_{hypersphere\ r} \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\left({y_1}^2+{y_2}^2+..+{y_n}^2\right)}$$

This integration can be performed by recognizing a similarity with the χ^2 distribution. Indeed, the χ^2 distribution corresponds to the distribution of the sum of the square of n random variable following a standard normal distribution.

$$\|y\|^2 \sim \chi^2(r, 2r)$$

$$\|y\| \sim \chi^2\left(\sqrt{r}, 2\sqrt{r}\right)$$

The distribution function for a given distance from the origin of the normalized space is the χ^2 distribution with n degrees of freedom, a mean value equals to $\sqrt{r}$ and a variance equals to $2\sqrt{r}$.

4.3.4 Probability of failure of a single linear limit state. It has been seen that the reliability index defines the level of safety of a linear limit state. The addition to a distribution function to the input variable $\boldsymbol{x}$ enables to evaluate the distribution function for $\boldsymbol{y}$ and for any distance from the origin in the normalized space. Thus, the probability of being in the failure set could be evaluated by an integration of the density function of $\boldsymbol{x}$ on the failure set as follows:

$$P(M(\boldsymbol{x}) \leq 0) = \int_{M \leq 0} p\Big(\boldsymbol{x};\ \boldsymbol{E}(\boldsymbol{x}), \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T})\Big) \cdot dx_1 \cdots dx_n$$

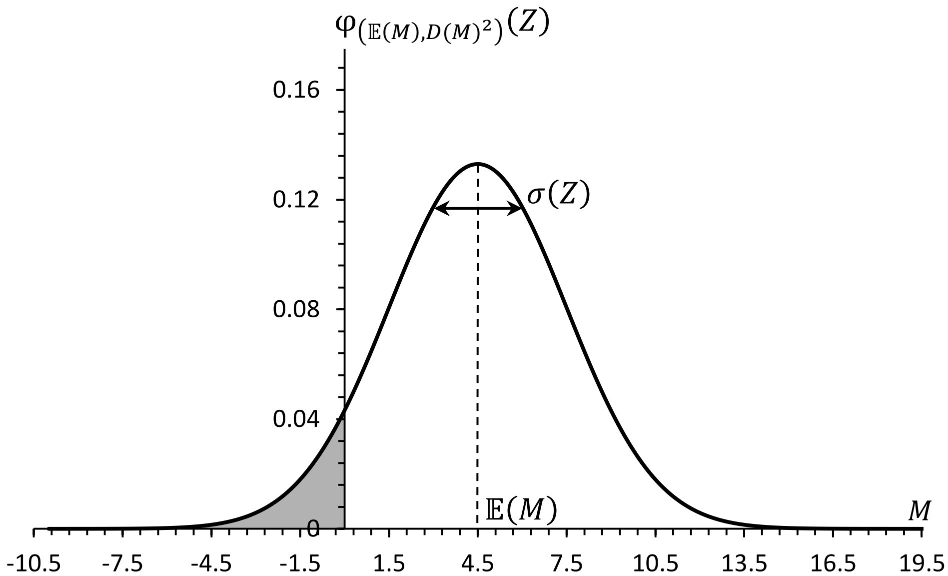
$$P(M(\boldsymbol{x}) \leq 0) = \int_{M \leq 0} \frac{1}{(2\pi)^{\frac{n}{2}} \cdot \sqrt{\boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T})}} \cdot e^{\left(-\frac{1}{2} \cdot (\boldsymbol{x} - \boldsymbol{E}(\boldsymbol{x}))^T \cdot \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T})^{-\boldsymbol{1}} \cdot (\boldsymbol{x} - \boldsymbol{E}(\boldsymbol{x}))\right)} \cdot dx_1 \cdots dx_n$$

$$P(M(\boldsymbol{x}) \leq 0) = \int_{M \leq 0} p(\boldsymbol{y};\ \boldsymbol{0}, \boldsymbol{I}) \cdot dy_1 \cdots dy_n$$

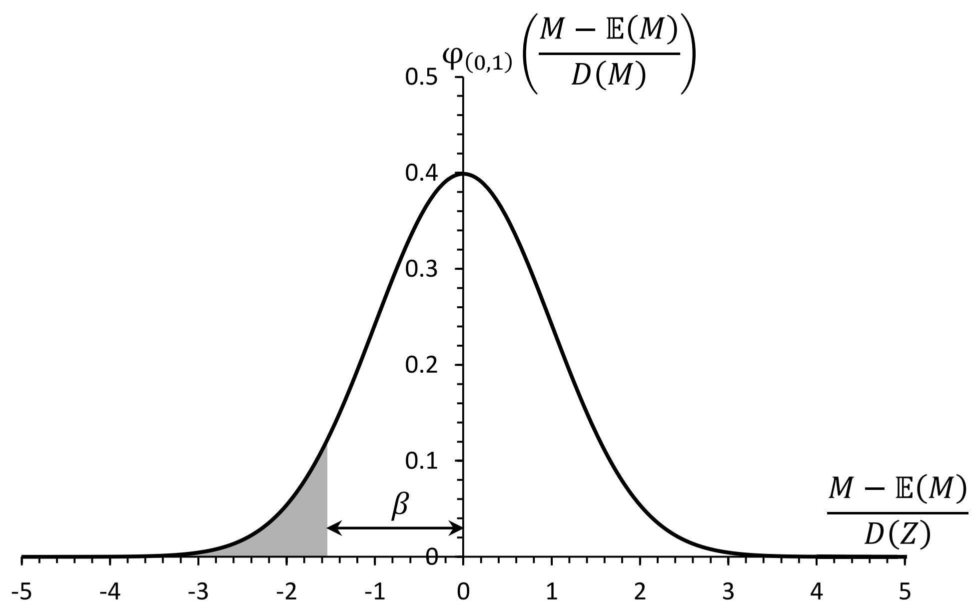
$$P(M(\boldsymbol{x}) \leq 0) = \int_{M \leq 0} \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\left({y_1}^2 + {y_2}^2 + .. + {y_n}^2\right)} \cdot dy_1 \cdots dy_n$$

The result of these integrals would be the exact probability of failure given a linear limit state. Although, the calculation of this integral is not really complicated, it has never used in the literature, as an approximation of the probability of failure based on the reliability index is preferred.

According to its definition, the safety index measures the distance from the limit state represented by $M = 0$ to the mean value $E(M)$ with the standard deviation $D(M)$ taken as a unit. Assuming that the safety margin M follows a normal distribution, $\frac{M-E(M)}{D(M)}$ follows a standard normal distribution. The probability of $M \leq 0$ can be computed using β as follows (see Figure 4.8):



(a) Assumed distribution function for the safety margin



(b) Reliability index

Figure 4.8 Reliability index β and safety margin distribution

Thus, the probability of failure is:

$$P(M(\boldsymbol{x}) \leq 0) = \Phi(-\beta) = 1 - \Phi(\beta)$$

In which:

- Φ is the standard normal cumulative distribution function.

In the case of the probability of survival, the outcome of the safety margin is in the safe set, that is $M(\boldsymbol{x}) > 0$. Thus, the probability of survival is:

$$P(M(\boldsymbol{x}) > 0) = \Phi(\beta)$$

4.4 The load continuity hypothesis

Before, introducing the load continuity hypothesis, the context in which it takes place must be explained. As explained later, the limit state governing a member failure can be approximated as linear limit state. In this approximation, the input variables representing the structure geometry and materials are considered as constant and only input variables which are considered to be random are the loads. In this approximation the vector $\boldsymbol{x}$ describes solely the loading.

4.4.1 Introduction of a loading evolution function. In the development of the limit state theory, independent variables are chosen to describe the structure and its loading. The purpose of these variables is to form a mathematical object on which limit states equation can be formulated. However, this object is purely mathematical and may in some case represent an impossible situation. A simple example can be given as the model describing a loading that is larger than the carrying capacity of the structure.

As reported in the literature, when randomness is associated to the input variables, the occurrence of the impossible situation corresponding to the input vector $\boldsymbol{x}$ in the failure set is taken as the occurrence of failure. Indeed, it was explained, that the probability of failure with regard to a limit state is the probability of the safety margin to be negative, which is the probability of the input vector $\boldsymbol{x}$ to overpass the limit state which correspond to a physically impossible situation.

Furthermore, limit states are approximated to linear limit state by considering only the loads as random variables. Given this description, an impossible situation is the occurrence of a loading which has overpassed the limit state. This situation is physically impossible because the physical nature of loads does not enable them to be suddenly equal to a given value. The evolution of the loading in time is continuous. This motivates the introduction of a loading evolution function which represents for any time the value of the loading. This parametric function is continuous on its domain.

At the first glance, the definition of a loading evolution function may appear contrary to the definition of random loading, which basically considers that any load can suddenly occur with an associated probability of occurrence. These two concepts are compatible; however, for any possible outcome of the random loading, the loading evolution function can be plotted from the origin of the space to the desired outcome. Thus, the final loading, which is the loading given by the loading evolution function for its largest application time, corresponds to the outcome of the random loading (see Fig. 4.9).

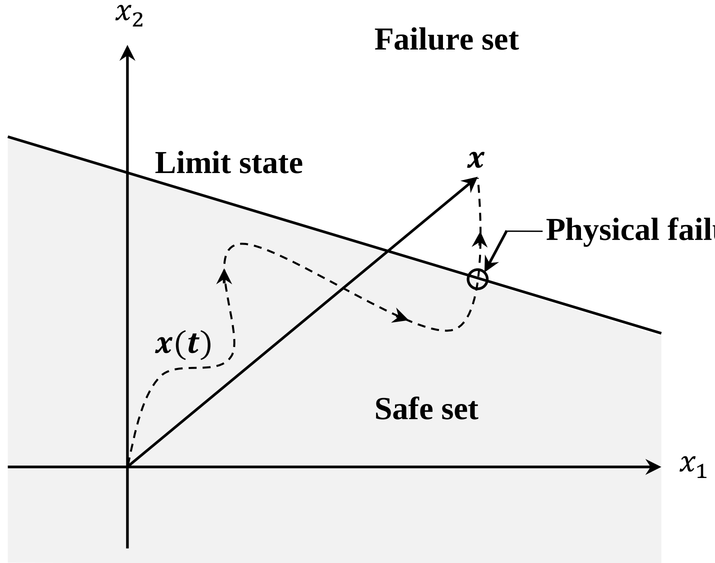


Figure 4.9 Loading evolution function in the subset $\mathfrak{L}$

As sown in Fig. 4.9, the evolution function $\boldsymbol{x(t)}$ can be with the following properties:

- $\boldsymbol{x(t)}$ is continuous
- $\boldsymbol{x}\left(\boldsymbol{t_{final}}\right) = \boldsymbol{x}$

Moreover, it is seen that for any $\boldsymbol{x}$ in the failure set, the physical failure with respect to the limit state has occurred. Thus, the probability of having $\boldsymbol{x}$ in the failure set, although being an imaginary situation, corresponds to the probability of failure.

$$P(M(\boldsymbol{x}) \leq 0) = P(\exists\ t\ |M\left(\boldsymbol{x}(\boldsymbol{t})\right) = 0)$$

Where:

- $P(M(\boldsymbol{x}) \leq 0)$ is calculated using the linear state theory described above.

- $\boldsymbol{x}(\boldsymbol{t})$ is the load evolution function
- $P(\exists\ t\ |M(\boldsymbol{x}(\boldsymbol{t})) = 0)$ is a definition of the probability of failure consistent with the load continuity.

This description of the continuity of the loading will be more significant in the case of multiple linear limit state and when it will be needed to ensure the continuity of the load in order to compute a second failure in the structure.

4.4.2 Elimination of inconsistent limit states. In the case of a multiple limit space, it is possible that, for a given outcome of the input variable vector $\boldsymbol{x}$, the load evolution function crosses to limit state hyperplanes. This case corresponds to no physical reality, since overpassing one limit state induces the failure of the structure. However, if this situation is implemented without modification in a reliability model, a probability may be assigned to the occurrence of being in the failure set which is completely included in another failure set. Figure 4.10 depicts a situation where multiple limit states are considered. The failure set corresponding to some of them can only be reached if the load evolution function has crossed another limit state hyperplane first. In order to make the reliability model more consistent with the reality, these particular limit states should be removed from the model. Figure 4.11 shows the reliability model which should be considered for this case.

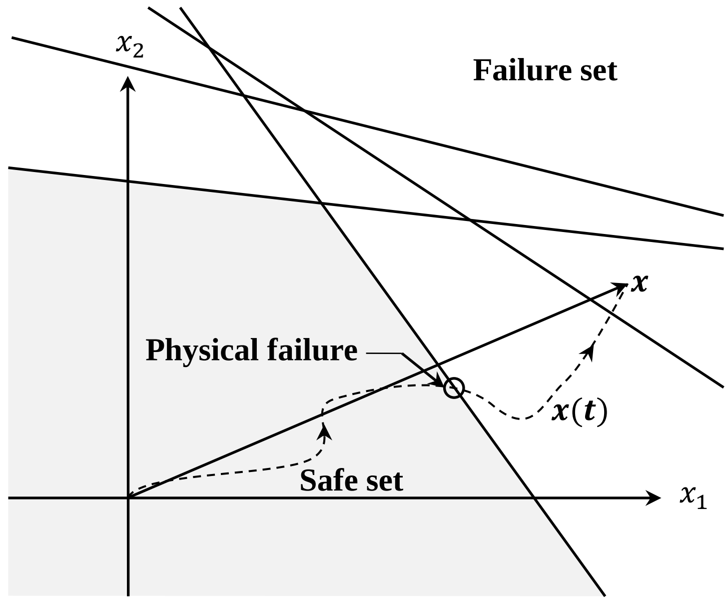


Figure 4.10 Reliability model with inconsistent limit states

Figure 4.10 has two limit states and has no chance to occur, given the continuity of the load evolution function. These inconsistent limit states are removed in the reliability model used to perform the reliability analyses depicted by Fig. 4.11.

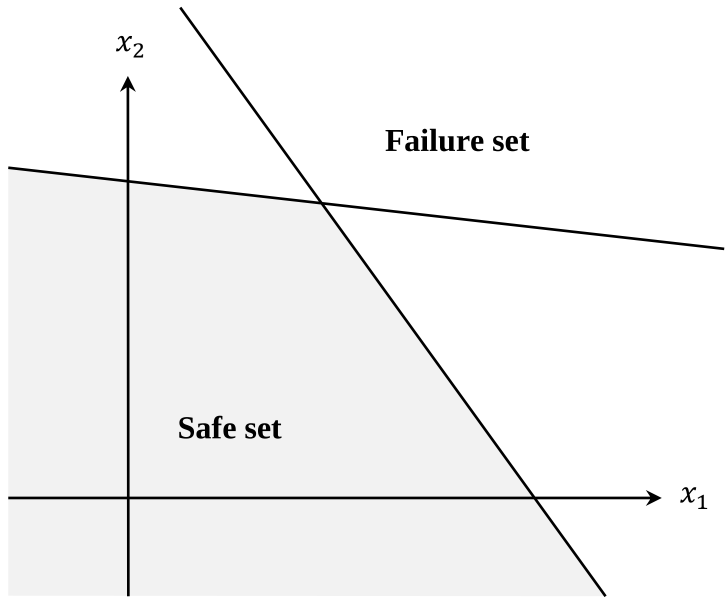


Figure 4.11 Corrected reliability model without inconsistent limit states

4.4.3 Elimination of unnecessary limit states. In order to accelerate the reliability analyses, a criterion to eliminate any unnecessary limit state can be established. The probability associated with the failure set of an unnecessary limit state is very low and can be neglected. To do so, a criterion must be defined to detect unnecessary limit states. This criterion is arbitrary. For example, using the reliability indices calculated with the single linear limit state theory, a criterion for unnecessary limit states could be to have a reliability index larger than a number of times the smallest reliability index calculated. Thus, if β_i is the reliability index associated with the limit state i, this limit state is considered as unnecessary if $\beta_i > \alpha \cdot \min_j(\beta_j)$.

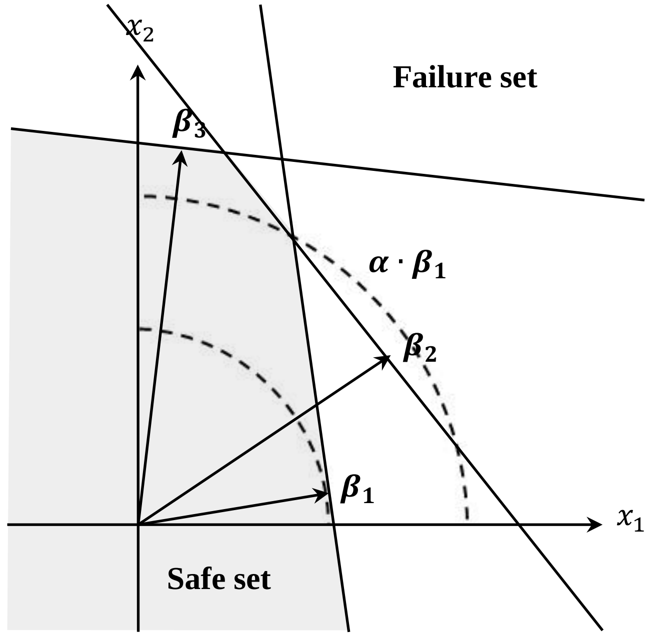


Figure 4.12 Reliability model with unnecessary limit states

Figure 4.12 has one limit state whose reliability index is larger than the smallest reliability index times a given coefficient $\boldsymbol{\alpha}$. This limit state corresponds to very few probabilities and can be neglected in order to speed up the reliabilities analyses. Figure 4.13 shows a corrected reliability model after removal of unnecessary limit states.

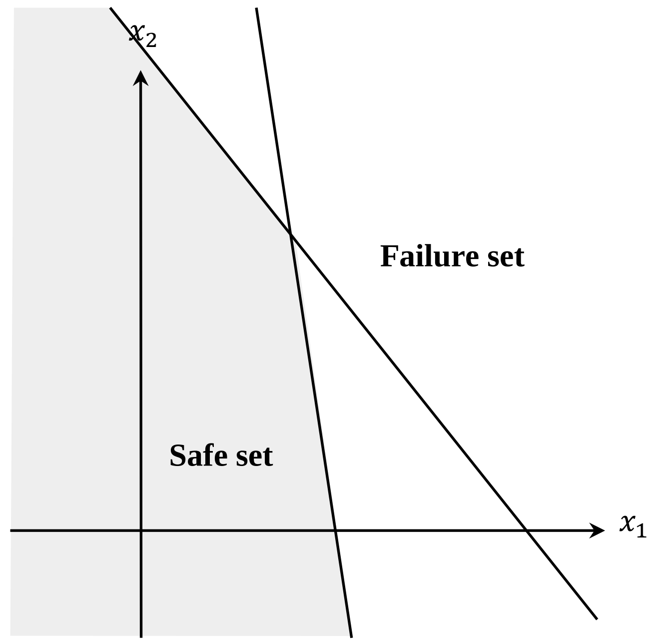


Figure 4.13 Corrected reliability model upon removing unnecessary limit states

4.4.4 The post failure continuity of the load. As previously explained in this chapter, the load continuity function is introduced to take into account the continuity of the load. This function is continuous at any given time and has no reason to be discontinuous when a failure occurs in the structure. Indeed, in reality, the loading of a structure does not depend of the behavior of the structure. However, in the classical methods presented earlier, the failure of a member occurs by an outcome of the random loading vector $\boldsymbol{x}$ in the failure set. This given outcome is not considered as a candidate for the failure of a second member, since it is not considered any longer once a first failure has occurred. Thus another outcome of the random vector $\boldsymbol{x}$ in the new failure set is required to cause the failure of a second member. This process forces the function representing the load evolution to be discontinuous when a failure occurs within the structure, which is not a realistic description of the reality and does not agree with the load continuity hypothesis. Moreover, since the random characteristics of the random load vector $\boldsymbol{x}$ are not updated after a failure, the expected value of the next outcome of $\boldsymbol{x}$, to be computed after a failure, is the expected value of the initial vector $\boldsymbol{x}$ which is often, conservatively, located far from the failure set. Thus, the transition between an outcome in the failure set and the next outcome to be computed is not continuous and gives additional chances of survival to the structure with regard to a second failure.

In reality, the load has no reason to change instantly because of a failure and the outcome of the vector $\boldsymbol{x}$ should be considered as a candidate for multiple failures. However, this cannot be done since the probabilistic theories introduced before do not enable us to know the outcome of $\boldsymbol{x}$, which has caused the failure. Nevertheless, the distribution function for $\boldsymbol{x}$ is known and the most probable outcome given a particular failure can be computed. Thus, as an approximation of the loading continuity, the variable which should be used to check if any more failures occur within the structure should be the most probable outcome given the occurrence of a particular failure. With consistency with the random variable $\boldsymbol{x}$, and by recognizing that the actual outcome of $\boldsymbol{x}$ may be different than the most probable outcome of $\boldsymbol{x}$ given the occurrence of a particular failure, a new random variable should be introduced. This random variable will have an expected value equal to the most probable outcome of $\boldsymbol{x}$ given the occurrence of a particular failure. Furthermore, its variance equals the residual variance between $\boldsymbol{x}$ and the most probable outcome of $\boldsymbol{x}$, which is the variance of the distance between $\boldsymbol{x}$ and the most probable outcome of $\boldsymbol{x}$. Thus the random variable $\boldsymbol{x}$ is updated to compute a second failure; and a new second moment description is assigned to $\boldsymbol{x}$ when a failure has occurred.

$$\boldsymbol{x} \to \boldsymbol{E}(\boldsymbol{x}|\boldsymbol{M_i}(\boldsymbol{x}) \leq \boldsymbol{0}) = \begin{pmatrix} E(x_1|M_i(x) \leq 0) \\ E(x_2|M_i(x) \leq 0) \\ \vdots \\ E(x_n|M_i(x) \leq 0) \end{pmatrix}$$

$$\boldsymbol{x} \to \boldsymbol{Cov}((\boldsymbol{x} - \boldsymbol{E}(\boldsymbol{x}|\boldsymbol{M_i}(\boldsymbol{x}) \leq \boldsymbol{0})), (\boldsymbol{x} - \boldsymbol{E}(\boldsymbol{x}|\boldsymbol{M_i}(\boldsymbol{x}) \leq \boldsymbol{0}))^{\boldsymbol{T}})$$

4.5 The most central failure point

4.5.1 Case of a single linear limit state (adopted from [2]). The most central failure point is defined as the most probable outcome of $\boldsymbol{x}$ given the occurrence of a particular failure. In the case of a single linear limit state, it can be written as follows:

$$\boldsymbol{E}(\boldsymbol{x}|M(\boldsymbol{x}) \leq 0)$$

This value can be found by integration of the failure set of the density function of $\boldsymbol{x}$. However, a close form can be given for $\boldsymbol{E}(\boldsymbol{x}|M(\boldsymbol{x}) = 0)$. Hereafter the following approximation will be considered: $\boldsymbol{E}\ (\boldsymbol{x}|M(\boldsymbol{x}) \leq 0) = \boldsymbol{E}(\boldsymbol{x}|M(\boldsymbol{x}) = 0)$.

A linear regression of $\boldsymbol{x}$ on the hyperplane $M(x) = 0$ is given as follows:

$$\widehat{\boldsymbol{E}}(\boldsymbol{x}|M) = \boldsymbol{E}(\boldsymbol{x}) + \frac{Cov(\boldsymbol{x}, M)}{Var(M)} \cdot \left(M - E(M)\right)$$

$$\widehat{\boldsymbol{E}}(\boldsymbol{x}|M = 0) = \boldsymbol{E}(\boldsymbol{x}) - \frac{Cov(\boldsymbol{x}, M)}{Var(M)} \cdot E(M)$$

$$\widehat{\boldsymbol{E}}(\boldsymbol{x}|M = 0) = \boldsymbol{E}(\boldsymbol{x}) - \frac{Cov(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{T}})}{\sqrt{a^T \cdot Cov(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{T}}) \cdot a}} \cdot a \cdot \beta$$

And in the normalized space, $\boldsymbol{E}(\boldsymbol{x}) = 0$ and $Cov(\boldsymbol{x}, \boldsymbol{x^T})$, so:

$$\widehat{\boldsymbol{E}}(\boldsymbol{x}|M = 0) = \beta \cdot \frac{a}{\sqrt{a^T \cdot a}}$$

This is the projection point of the origin of the normalized space on the hyperplane given by $M = 0$.

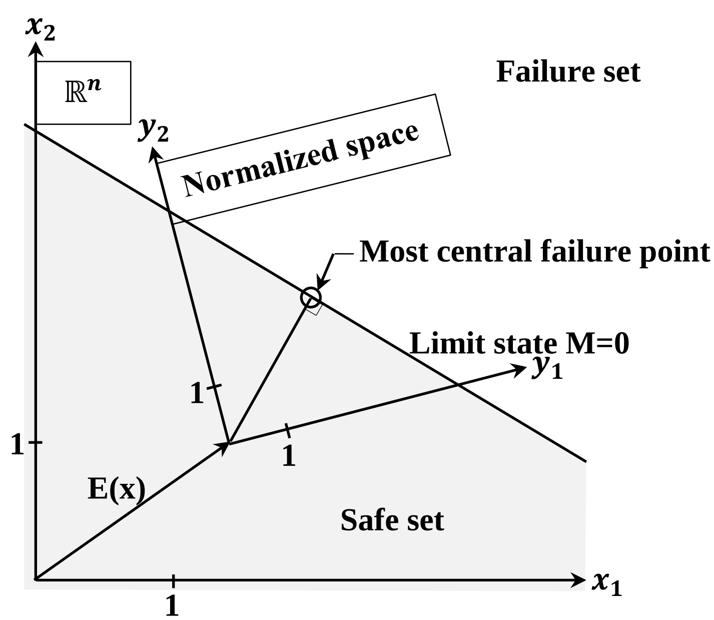


Figure 4.14 Most central failure point

4.5.2 Case of multiple limit states. The reliability model introduced in this study does not enable to find the most probable outcome of the vector $\boldsymbol{x}$ given that a particular failure has occurred. Even though the most central failure point for a failure set defined by multiple linear limit state could be found by integration of the density of $\boldsymbol{x}$ on the failure set; among multiple linear limit state it is not possible for to locate the most central failure point for each linear limit state.

In the case of multiple limit states, the most probable outcome of the vector $\boldsymbol{x}$ given that a particular failure has occurred must be approximated using the case of single limit state taken apart. Thus, the results of the previous section can be used for multiple limit states. However, there are some cases where using this approximation may lead to inconsistency. When the most central failure point of the single limit state (taken apart) is not on the surface that represents the safe, a failure set with multiple linear limit state is considered. Figure 4.15 depicts this situation. In these cases, the most central failure point is chosen as the closest point, which is on the surface describing the safe and the failure set.

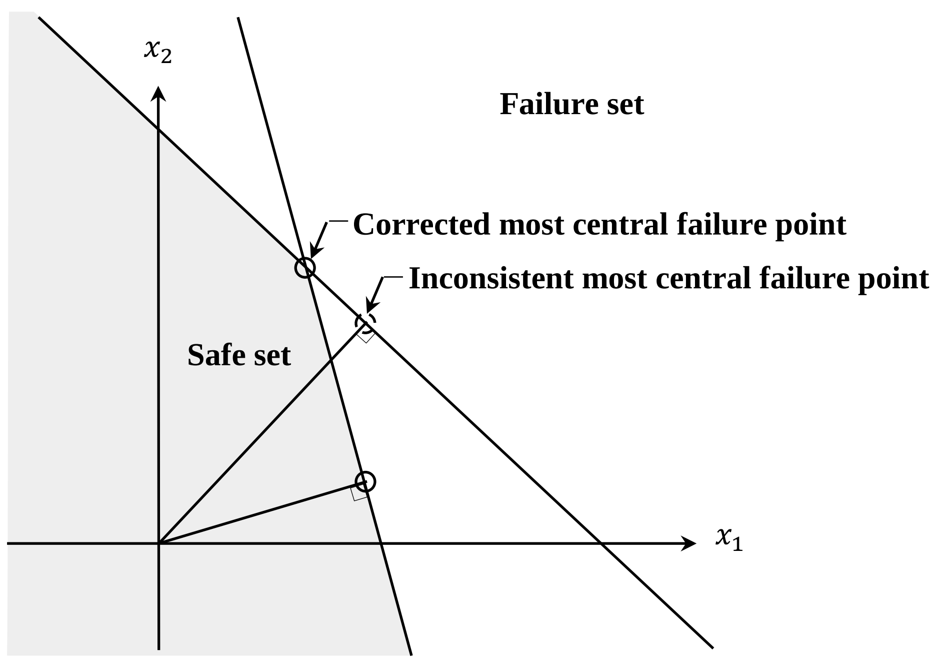


Figure 4.15 Most central failure points for multiple limit states

4.6 Probability calculations for multiple linear limit states

4.6.1 Extension of the single linear state. A definition of the safe set for multiple has been given earlier as:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \mid \forall\, i, f_i(\boldsymbol{x}) > 0\,\} = \bigcap_i \{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \ \mid f_i(\boldsymbol{x}) > 0\,\}$$

The probability of the occurrence of an outcome of the vector $\boldsymbol{x}$ in the safe set can be calculated as follows:

$$P(\forall\, i, f_i(\boldsymbol{x}) > 0\,) = \prod P(f_i(\boldsymbol{x}) > 0 \ \mid \forall j < i\; f_j(\boldsymbol{x}) > 0\,)$$

For the failure set a similar definition can be given:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \mid \exists\, i, f_i(\boldsymbol{x}) \leq 0\,\} = \bigcup_i \{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n \ \mid f_i(\boldsymbol{x}) \leq 0\,\}$$

The probability of the occurrence of an outcome of the vector $\boldsymbol{x}$ in the failure set can then be calculated as follows:

$$P(\exists\, i, f_i(\boldsymbol{x}) \leq 0\,) = \sum_i P(f_i(\boldsymbol{x}) \leq 0\,) - \sum_{i,j} P\big(f_i(\boldsymbol{x}) \leq 0 \ \cap P(f_j(\boldsymbol{x}) \leq 0\big) + \sum_{i,j,k} \cdots$$

Those two previous probabilities are difficult to evaluate since, the linear limit state theory using the reliability index is only able to evaluate the probability of being in the failure set when limit state are taken apart.

To circumvent the difficulty encountered when evaluating the probability of the outcome of $\boldsymbol{x}$ in several failure sets, the following solution can be used. Since it was seen that the probability of failure with regard to a single limit state corresponds to the probability to have the load evolution function crossing the limit state equation, the probability to have an outcome of $\boldsymbol{x}$ located exactly on the limit state is taken as the probability to have an outcome of $\boldsymbol{x}$ in the failure set.

$$P(M_i(\boldsymbol{x}) = 0) = P(M_i(\boldsymbol{x}) \leq 0)$$

The same solution is used for a multiple linear limit state. The probability of the outcome of the vector $\boldsymbol{x}$ to be located on the surface representing the safe set is taken as equal to the probability of the outcome of the vector $\boldsymbol{x}$ to be in the failure set for the multiple linear limit state.

$$P(\exists\, i, M_i(\boldsymbol{x}) = 0 \cap \forall\, k \neq i\ M_k(\boldsymbol{x}) > 0) = P(\exists\, i, M_i(\boldsymbol{x}) \leq 0\ )$$

This set $\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n | \exists\, i, M_i(\boldsymbol{x}) = 0 \cap\ \forall\, k \neq i\ M_k(\boldsymbol{x}) > 0\}$ can be partitioned into several mutually exclusive sets for a given i as follows:

$$\{\boldsymbol{x} \in \mathfrak{L} \subset \mathbb{R}^n |\ M_i(\boldsymbol{x}) = 0 \cap\ \forall\, k \neq i\ M_k(\boldsymbol{x}) > 0\}$$

Thus, the probability of failure can be calculated with a simple summation:

$$P(\exists\, i, M_i(\boldsymbol{x}) = 0 \cap\ \forall\, k \neq i\ M_k(\boldsymbol{x}) > 0) = \sum_i P(M_i(\boldsymbol{x}) = 0 \cap\ \forall\, k \neq i\ M_k(\boldsymbol{x}) > 0)$$

For the sake of simplicity $P(M_i(\boldsymbol{x}) = 0 \cap\ \forall\, k \neq i\ M_k(\boldsymbol{x}) > 0)$ will be denoted as P_f and can be rewritten as:

$$P_f = \sum_i P(M_i(\boldsymbol{x}) = 0) \cdot P(\forall\, k \neq i\ M_k(\boldsymbol{x}) > 0\ |(M_i(\boldsymbol{x}) = 0)$$

It has been shown that:

$$P(M_i(\boldsymbol{x}) = 0) = 1 - \Phi(\beta_i)$$

And the probability $P(\forall\, k \neq i\; M_k(\boldsymbol{x}) > 0 \,|(M_i(\boldsymbol{x}) = 0)$ can be seen as a new safe set situation where the vector $\boldsymbol{x}$ is constraint on the hyperplanes defined by $(M_i(\boldsymbol{x}) = 0)$ and must be in the safe set of the other limit state hyperplanes. The distribution associated with the vector $\boldsymbol{x}$ on a given hyperplane will be demonstrated later in this thesis. Moreover, the method consisting in integrating the distribution on a defined safe set will be explained in the next paragraph. However, before any further explanations, this probability can be shown to be as follows:

$$P(\forall\, k \neq i\; M_k(\boldsymbol{x}) > 0 \,|(M_i(\boldsymbol{x}) = 0) = \int\limits_{y\,\epsilon\,\mathcal{P}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}\|y - y_0\|^2}$$

Where:

- $\mathcal{P}$ is a bounded domain of the n-1 dimensional hyperplane defined by $f_i(\boldsymbol{x}) = 0$.
- $\mathcal{P}$ is bounded by the conditions $f_k(\boldsymbol{x}) > 0$.
- y_0 is the most central point for the limit state i. This notion will be explained later in this study.

Or,

$$P(\forall\, k \neq i, f_k(\boldsymbol{x}) > 0 \mid f_i(\boldsymbol{x}) = 0) = \int\limits_{y\,\epsilon\,\mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$$

- $\mathcal{R}$ is a subset of $\mathbb{R}^{n-1}$
- $\mathcal{R}$ is bounded by the distances, calculated on the hyperplane defined by $f_i(\boldsymbol{x}) = 0$, between y_0 and $f_k(\boldsymbol{x}) = 0$

Then, the final probability failure is:

$$P_f = \sum_{i=1}^{m} (1 - \phi_n(\beta_i)) \cdot \int_{y \in \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$$

And the probability failure with regard to the linear limit state i among multiple linear limit state is a term of the above summation:

$$P_{fi} = (1 - \phi_n(\beta_i)) \cdot \int_{y \in \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$$

Owing to the solutions consisting of dividing the failure set into subset, it is possible to compute the probability of failure using the mutually exclusive events through a simple summation. The events of member failures are assumed mutually exclusive since P_{fi} can be calculated on domain which is not intersected by any other domain corresponding to the failure of another member. However, it should be remembered that this solution is an approximation. Moreover, the final expression of the probability of failure for multiple linear limit states shows the complexity of the problem. Indeed, in the final expression of the probability of failure, the integral on the subset $\mathcal{R}$, which refers to the probability of $\boldsymbol{x}$ to be in the safe set for every remaining limit state while being located on the limit state i, can be calculated with a similar method. This recursive process continues until the dimension of the subset equals 0 where no integration can be conducted or when a subset which is not represented by any other limit state is encountered. This second case occurs when the number of limit state is less than the number of input variables.

In conducting the solution, the number of calculations needed will increase in a factorial manner proportional with the number of variables.

4.6.2 The simple integration. The simple integration method consists of integrating the probability distribution function on the failure set. The probability distribution function in the normalized space for the input variables which corresponds to the loading only is the standard normal multivariate function. The failure set corresponds to the union of the failure set defined for a single linear limit state. Because the failure set can be complex the integration must be performed numerically with a computer.

4.6.3 Monte Carlo simulation. A Monte Carlo simulation would consist in generating random variables $\boldsymbol{y}$ following standard normal multivariate distribution and to compute the probability of being in the failure set by counting the number of outcomes of $\boldsymbol{y}$ which are in the failure set of the normalized space.

In this case, the algorithm could also compute the probability that a given limit state is responsible for the failure by assuming that the load evolution function for every outcome of the random vector $\boldsymbol{y}$ is a straight line in the normalized space and by counting the number of outcomes of $\boldsymbol{y}$ with a load evolution function crossing the considered limit state hyperplane before crossing any other hyperplane.

Figures 4.16 and 4.17 show how the surface representing the safe and the failure set is shaped. Figure 4.17 shows the load evolution function, which is considered to be a straight line. As seen in this figure, the surface defining the safe and the failure set is shaped using straight line segments. In that way, the load evolution function can cross only one limit state, which corresponds to the limit state responsible for the failure.

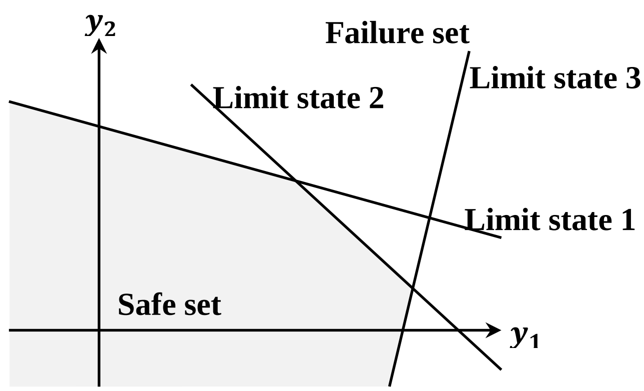


Figure 4.16 Multiple linear limit states before shaping

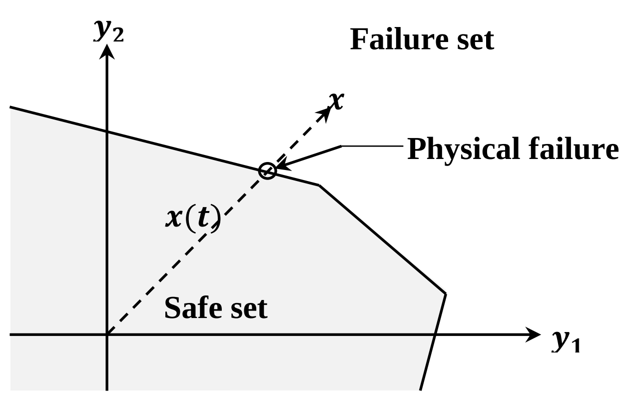


Figure 4.17 Multiple linear limit states after shaping and straight load evolution function

4.6.4 The simple failure probability bounds. The probability of failure with regard to multiple limit states can be bounded using the probabilities of single limit state taken apart as it was explained in the linear limit state theory. The first bound considers that all the lower states are perfectly correlated and in that case:

- $p_f = \max(p_{fi})$
- p_{fi} is the probability of failure of the single linear limit states taken apart

The upper bound considers that all the limit states are independent and in that case:

- $p_f = \sum_{i=1}^{m} p_{fi}$

Note that in latter equation, the failure probabilities are extremely small. Otherwise,

- $p_f = 1 - \sum_{i=1}^{m} (1 - p_{fi})$

In the normalized space, a representation of the limit states hyperplanes corresponding to these two situations can be demonstrated. In the case of perfectly correlated limit states, the hyperplanes are all parallel and in the case of independent the hyperplanes are orthogonal with respect to each other (figure 4.18 and 4.19).

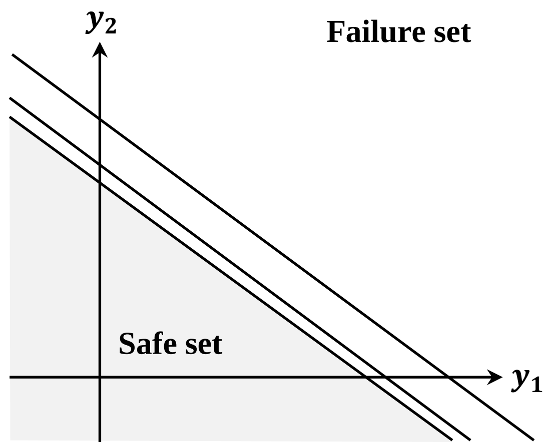


Figure 4.18 Perfectly correlated limit states

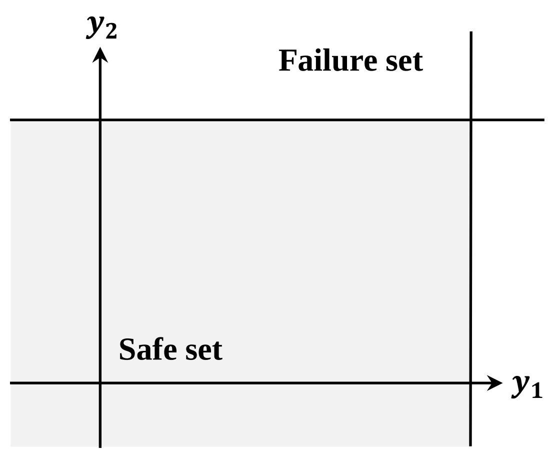


Figure 4.19 Independent limit states

In real situations, the limit states are neither perfectly correlated nor independent; the probability of failure is therefore bounded as follows:

$$\max(p_{fi}) \leq p_f \leq \sum_{i=1}^{m} p_{fi}$$

4.6.5 The E.G. Kounias bounds. Usually, to compute the probability of failure of multiple linear limit state, the simple bounds are used. However, in some cases these simple bounds are rather wide. This motivated the formulation of more precise bounds by the statistician E.G. Kounias [2]. This method is capable of providing a more accurate estimation of the probability of failure by going a step further. The method proposes to include in the bounds the influence of the probability of failure of two possible limit states. In that way, the simple bounds approximation made in considering only single limit state apart is corrected by evaluating the influence of the other limit states and especially the probability of being in the failure set of a first single limit state taken apart while being in the failure set of a second single limit state taken apart. Thus probabilities of being in two failure set are the computed.

An exact computation of the probability of failure would also evaluate this influence with regard to the other remaining limit states. This would result in a recursive process as previous described. According to E.G. Kounias:

$$P(\mathcal{F})\begin{cases} \geq P(\mathcal{F}_1) + \sum_{i=2}^{m} max\left\{P(\mathcal{F}_i) - \sum_{j=1}^{i-1} P\left(\mathcal{F}_i \cap \mathcal{F}_j\right), 0\right\} \\ \leq \sum_{i=1}^{m} P(\mathcal{F}_i) - \sum_{i=2}^{m} max_{j<i}\left\{P\left(\mathcal{F}_i \cap \mathcal{F}_j\right)\right\} \end{cases}$$

Where:

- $P(\mathcal{F}_i) = P(M_i \leq 0)$ is the probability of failure of the limit state i taken as a single limit state.
- $P\left(\mathcal{F}_i \cap \mathcal{F}_j\right) = P\left(M_i \leq 0 \cap M_j \leq 0\right)$ is the probability of failure from both events.

The probability $P(\mathcal{F}_i \cap \mathcal{F}_j)$ can be calculated using the two-dimensional multivariate standard normal distribution as follows:

$$P(\mathcal{F}_i \cap \mathcal{F}_j) = \Phi_2(-\beta_i, \beta_j, \rho_{ij})$$

Where:

- ρ_{ij} is the coefficient of correlation of the two considered limit state
- Φ_2 is the two-dimensional multivariate standard normal distribution

The value of Φ_2 must be calculated numerically since no table exists for every ρ_{ij} for the two-dimensional multivariate standard normal distribution. However, this probability can be bounded as follows:

For $\rho_{ij} > 0$:

- $P(\mathcal{F}_i \cap \mathcal{F}_j) = \phi_2(-\beta_i, \beta_j, \rho_{ij}) \begin{cases} \geq max\{\phi(-\beta_i) \cdot \phi(-\beta_{j|i}), \phi(-\beta_j) \cdot \phi(-\beta_{i|j})\} \\ \leq \phi(-\beta_i) \cdot \phi(-\beta_{j|i}) + \phi(-\beta_j) \cdot \phi(-\beta_{i|j}) \end{cases}$

For $\rho_{ij} < 0$:

- $P(\mathcal{F}_i \cap \mathcal{F}_j) = \phi_2(-\beta_i, \beta_j, \rho_{ij}) \leq min\{\phi(-\beta_i) \cdot \phi(-\beta_{j|i}), \phi(-\beta_j) \cdot \phi(-\beta_{i|j})\}$

It can be seen in figure 4.20 that in the case of a positive coefficient of correlation, the probability evaluated by $\phi(-\beta_i) \cdot \phi(-\beta_{j|i})$ or by $\phi(-\beta_j) \cdot \phi(-\beta_{i|j})$ correspond to a set which is included in $\mathcal{F}_i \cap \mathcal{F}_j$. Then the maximum of these two probability is chosen as a lower bound for $P(\mathcal{F}_i \cap \mathcal{F}_j)$. The situation is the same in the case of a negative coefficient of correlation, but in this case, the computed probabilities are larger than the actual probability of being in the two failure set and the minimum must be computed.

Figure 4.20 shows the joint failure set for the two limit states i and state j. In this case ρ_{ij} is positive because the angle between the normal of the two limit states pointing toward the failure set is less than $\frac{\pi}{2}$. Figure 4.21 depicts the failure set corresponding to the probability $\phi(-\beta_i) \cdot \phi(-\beta_{j|i})$. It is seen that the approximated set is included in the actual failure set depicted in figure 4.20. and figure 4.22 depicts the failure set corresponding to the probability $\phi(-\beta_i) \cdot \phi(-\beta_{j|i})$. The approximated set in this figure is even smaller than the one depicted by Fig. 4.21. Thus, the best approximation for the lower bound of the actual probability of the joint failure set is the maximum of the two set depicted by figures. 4.21 and 4.22.

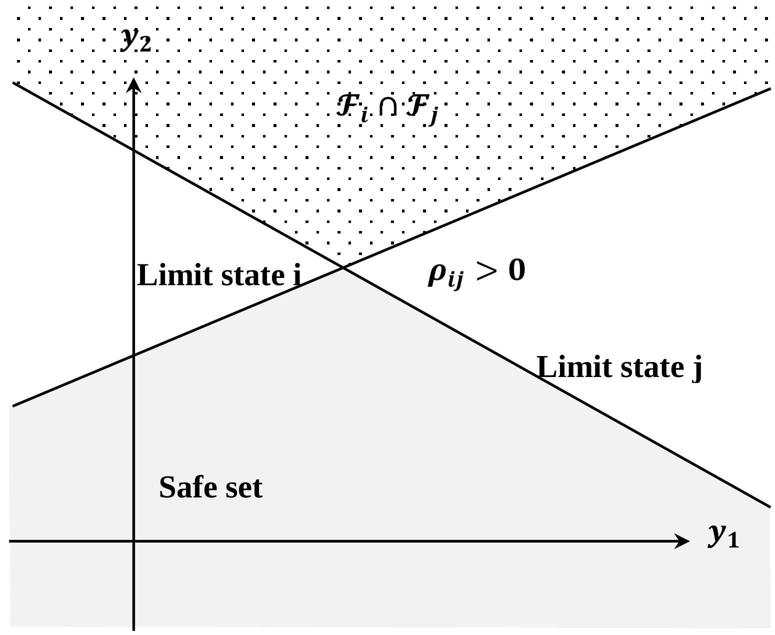


Figure 4.20 Joint failure set for the limit stat I and the limit state j

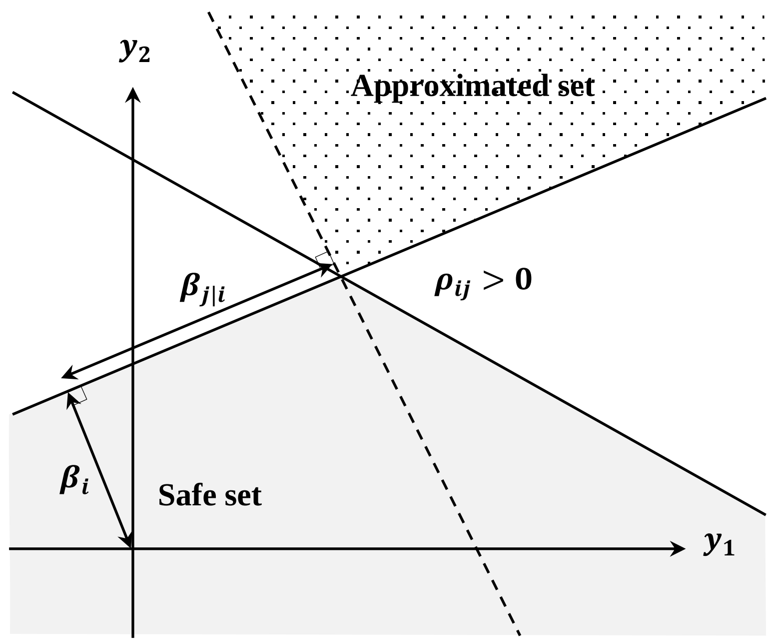


Figure 4.21 Failure set corresponding to the probability $\phi(-\beta_i) \cdot \phi\left(-\beta_{j|i}\right)$

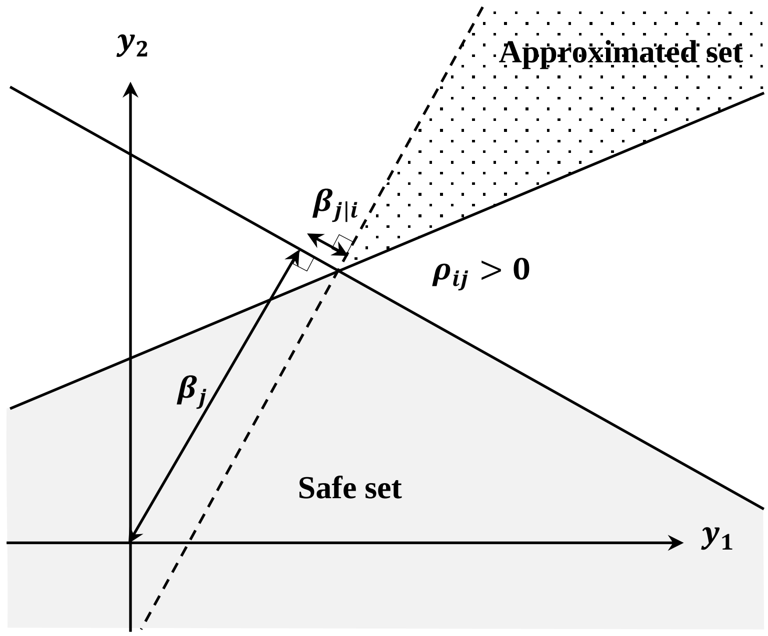


Figure 4.22 Failure set corresponding to the probability $\phi(-\beta_j) \cdot \phi(-\beta_{i|j})$

According to Ditlevsen and Madsen [2] the conditional reliability index can be calculated as follows:

$$\beta_{i|j} = \frac{\beta_i - \rho_{ij} \cdot \beta_j}{\sqrt{1 - {\rho_{ij}}^2}}$$

The coefficient of correlation between the limit states i and the limit stat j is written as follows:

$$\rho_{ij} = \frac{Cov(M_i, M_j)}{D(M_i) \cdot D(M_j)}$$

This coefficient of correlation can be interpreted in the normalized space as the cosine of the angle ν_{ij} between the two normal vectors of the two limit states hyperplanes pointing toward the failure set (see figure 4.23).

$$\rho_{ij} = \cos(\nu_{ij})$$

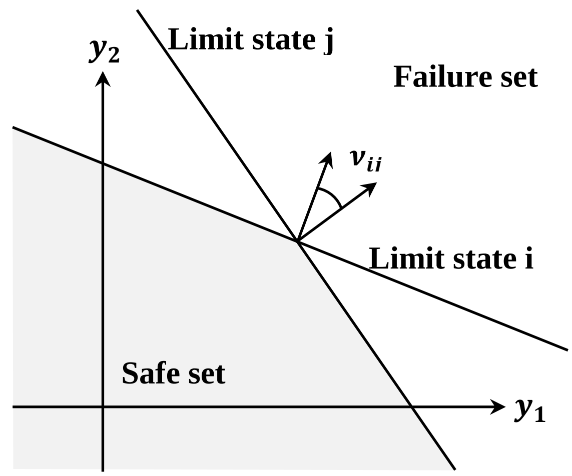


Figure 4.23 Geometric interpretation of the correlation coefficient ρ_{ij} [2]

The bounds associated with the geometric interpretation of the reliability indices provides an extra dimension in the process of the evaluation of the failure defined by multiple linear limit state. This method takes into consideration more data than the simple bounds method and as a consequence is more accurate in bounding the actual probability of failure.

4.6.6 The hybrid method. Most theories on the multiple linear limit state focus on the probability of failure with regard to all limit states. However, it is needed to know which limit state is responsible for the failure. Thus the probability of failure with regard to a given linear limit state i among multiple linear limit state must be computed. Previously, it was shown that this probability can be approximated as follows:

$$P_{fi} = (1 - \Phi(\beta_i)) \cdot \int_{y \,\epsilon\, \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$$

It was explained that the integral $\int_{y \,\epsilon\, \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$ corresponds to the probability of the outcome of the input variables to be located on the safe set for every limit state but i given that the outcome of the input variables is located on the failure set corresponding to the limit state i. Thus $1 - \int_{y \,\epsilon\, \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$ corresponds to the opposite probability which is the probability that the outcome of the input variables is located on the failure set for at least one limit state different than i given that the outcome of the input variables is located on the failure set corresponding to the limit state i. When only one limit state j is considered with the initial limit state i, this probability corresponds to the probability to be in the failure set defined by the limit state j given that the outcome of the input variables is in the failure set defined by the limit state i. This probability can be written as $P\left(\mathcal{F}_j | \mathcal{F}_i\right)$ and the term $\left(1 - \Phi(\beta_i)\right) = \Phi(-\beta_i)$ corresponds to the probability $P(\mathcal{F}_i)$. Thus,

$$P(\mathcal{F}_i)P\left(\mathcal{F}_j | \mathcal{F}_i\right) = P\left(\mathcal{F}_i \cap \mathcal{F}_j\right) = (1 - \Phi(\beta_i)) \cdot (1 - \int_{y \,\epsilon\, \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr)$$

Describing Kounias' bounds [2], given a coefficient of correlation, $P(\mathcal{F}_i \cap \mathcal{F}_j)$ is approximated in two different ways whether it is seen as $P(\mathcal{F}_i)P(\mathcal{F}_j|\mathcal{F}_i)$ or $P(\mathcal{F}_j)P(\mathcal{F}_i|\mathcal{F}_j)$.

$$P(\mathcal{F}_i)P(\mathcal{F}_j|\mathcal{F}_i) \cong \phi(-\beta_i) \cdot \phi(-\beta_{j|i})$$

$$P(\mathcal{F}_j)P(\mathcal{F}_i|\mathcal{F}_j) \cong \phi(-\beta_j) \cdot \phi(-\beta_{i|j})$$

In the P_{fi} formula, $(\mathcal{F}_i)$, since $P(\mathcal{F}_i)$ is already known, the former approximation in the two approximation given above is used. Thus, these two expressions can be given:

$$P(\mathcal{F}_i)P(\mathcal{F}_j|\mathcal{F}_i) == \Phi(-\beta_i) \cdot (1 - \int_{y \, \epsilon \, \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr)$$

$$P(\mathcal{F}_i)P(\mathcal{F}_j|\mathcal{F}_i) \cong \phi(-\beta_i) \cdot \phi(-\beta_{j|i})$$

It can be seen that, in the case where only one additional limit state j is considered, using the approximation made by Ditlevsen and Madsen [2] in his reliability index theory, it can be written that:

$$1 - \int_{y \, \epsilon \, \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr = \phi(-\beta_{j|i})$$

$$\int_{y \, \epsilon \, \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr = \phi(\beta_{j|i})$$

However, in real cases, more than one additional limit state must be considered. as shown previously, the simple bounds were computed to evaluate the influence of having multiple linear limit state when evaluating every limit state independently. Here, the case is equivalent, i.e., the influence of having multiple second linear limit state must be considered using probabilities computed for only one second linear limit state. Thus, similar bounds can be given:

$$\max(\phi(-\beta_{j|i})) \leq 1 - \int_{y\,\epsilon\,\mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr \leq \sum_{i=1}^{m} \phi(-\beta_{j|i})$$

$$\max\left(\phi(-\beta_{j|i})\right) - 1 \leq - \int_{y\,\epsilon\,\mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr \leq \sum_{i=1}^{m} \phi(-\beta_{j|i}) - 1$$

$$1 - \sum_{i=1}^{m} \phi(-\beta_{j|i}) \leq \int_{y\,\epsilon\,\mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr \leq 1 - \max\left(\phi(-\beta_{j|i})\right)$$

And finally, the probability of failure with regard to the linear limit state i among multiple linear limit state is bounded as follows:

$$P_{fi} = \Phi(-\beta_i) \cdot \int_{y\,\epsilon\,\mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$$

$$\Phi(-\beta_i) \cdot (1 - \sum_{i=1}^{m} \phi(-\beta_{j|i})) \leq P_{fi} \leq \Phi(-\beta_i) \cdot (1 - \max\left(\phi(-\beta_{j|i})\right))$$

4.7 Material post-failure behavior

It was explained that the over-strength of a structure can be provided by two types of redundancy. The first one, in a case of indeterminate structures, comes from the geometry of the structure whereas the second one exists in all structures and comes from the material. The material redundancy is defined by the material post-failure behavior. In Chapter 2 of this thesis, it was explained that three categories of material redundancy can be defined; namely, the capacity in withstanding more displacement, and the capacity in withstanding more constraints or a combination of both. Hereafter, it will be seen how to include the post-failure behavior of a material in the reliability model in order to take into account the material redundancy. It should be noted that the criterion for failure delimiting the pre-failure behavior and the post-failure behavior is the yielding.

4.7.1 Linear elastic – perfectly plastic model. In this model, the pre-failure behavior is considered as perfectly elastic and is compatible with a structure analysis such as the stiffness method. The criterion for failure is the yielding. The post-failure behavior is perfectly plastic, which means that no additional constraint after yielding can be withstood by the material. This model can be represented through the strain-stress constitutive relation curve as depicted in the Fig. 4.24. In this model, when a member has failed, its strength (the yield stress multiplied by the cross sectional area of the truss member) can be treated as a new force in the model. However, in the reliability model used in this thesis, the material characteristics are not taken as random and the force corresponding to a failed member is not including in the random vector $\boldsymbol{x}$.

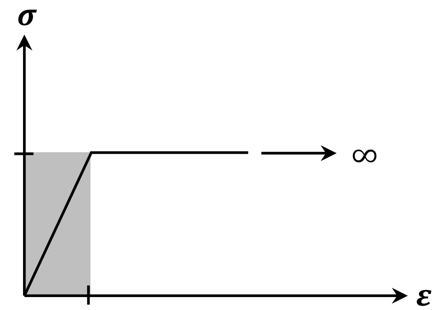


Figure 4.24 Linear elastic – perfectly plastic model

4.7.2 Linear elastic – linear work hardening model. In this model, the pre-failure behavior is considered as perfectly elastic and is compatible with a structure analysis such as the stiffness method. The criterion for failure is the yielding. The post-failure behavior is a linear work hardening which means that once the yielding is reached the post-failure relation between the stress and the strain is linear with a given young modulus E'. The linear elastic–linear work hardening model can be divided in two sub-models. The first one considers that the post failure has no limit in terms of admissible stress and strain whereas the second one considers a ductile failure for a given stress and strain. The two behaviors can be implemented in the same way in the reliability model.

The post-failure stress-strain relation can be written as follows:

$$\sigma = f_y + (\varepsilon - \varepsilon_y) \cdot E'$$

Where:

- f_y and ε_y are the stress and the strain corresponding to the yielding failure

- E' is the plasticity modulus

Usually, ε_y is small and can be neglected:

$$\sigma = f_y + \varepsilon \cdot E'$$

A similar relation can be written in term of force and displacement:

$$F = f_y \cdot A + \left(d - d_y\right) \cdot \frac{E' \cdot A}{L}$$

And neglecting d_y:

$$F = f_y \cdot A + d \cdot \frac{E' \cdot A}{L}$$

This relation can be implemented in post-failure structural analysis by considering in the stiffness method a new force equals to $f_y \cdot A$ at the location of the failed member and a new stiffness for the member equals to $\frac{E' \cdot A}{L}$.

$$F = F' + d \cdot k'$$

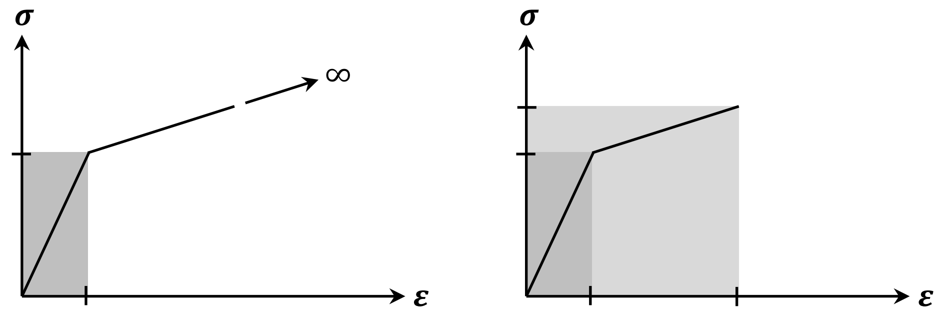
In the stiffness method, the elasticity modulus E, is usually not taken into account because it has no influence since it is usually common to all members. In order to compute the new stiffness k' for the member, a new member area could be considered in order to keep the same E.

$$k' = \frac{E' \cdot A}{L} = \frac{E \cdot A'}{L}$$

$$A' = \frac{E'}{E} \cdot A$$

In that way, the post-failure stiffness matrix can be easily computed. It should be noted that, because the displacement due to the member behavior before yielding is neglected the structure geometry does not change after a failure. To compute stiffness after a yielding failure, no additional structural analysis is needed since only the area of the failed member is modified using a fictive area A'. This is an important result because it enables to spare an important number of structural analyses.

The two cases can be implemented in the structural analyses the same way. However, depending on the behavior chosen a criterion for failure has to be selected. In the first case, which considers an infinite ductility capacity, the member never fails and its strength is theoretically capable of reaching any level. In that case, the stiffness matrix never gets singular and the occurrence of a mechanism within the structure is impossible. Thus, the classical criterion for failure cannot be used and another criterion must be defined which could be, for example, the deflection of a given point in the structure. Indeed, since the failure structure of the structure does not occurs with regard to the material and a geometric failure criterion must be defined. A criterion based on the deflection of a given based does not require any additional computation when using the presented stiffness. In the second case, which considers a ductile failure in the member, a second limit state for the member can be written to represent the ductile failure. This results in longer failure paths since each member fails twice. The figure 4.25(a) represents the case where the material has infinite ductility whereas the figure 4.25 (b) illustrates a material with a ductile failure.



(a) Infinite ductility (b) Ductile failure

Figure 4.25 Linear elastic – linear work hardening model

4.8 The linear limit state approximation

4.8.1 Introduction to convex sets. A convex set is a set which has the property that if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ belong to the safe set, for every t in the interval $[0;1]$, $(1-t)\boldsymbol{x}_1 + t\boldsymbol{x}_2$ belongs to the set. Convex safe sets are particularly convenient for reliability analysis because any limit state having a convex safe set can be described as the intersection, finite or infinite, of linear limit states. Thus a model including multiple linear limit states as presented before can approximate any convex sets and the methods described above can be used. Ditlevsen and Madsen in [2] provide two methods to approximate any convex sets to an intersection of multiple linear limit states depending on the curvature of the convex set (figures 4.26, 4. 27 and 4.28).

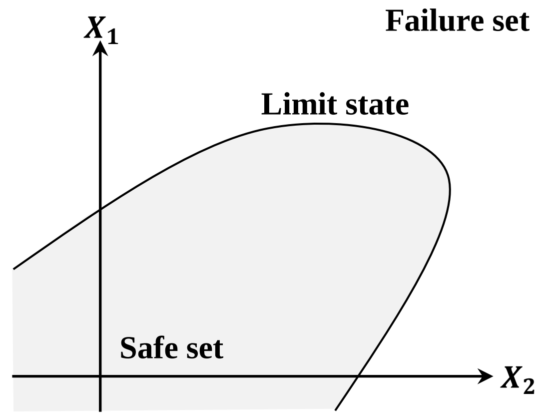


Figure 4.26 Convex safe set

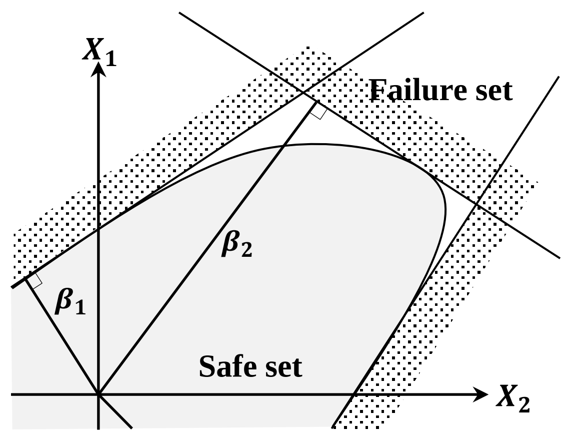


Figure 4.27 FORM approximation [2]

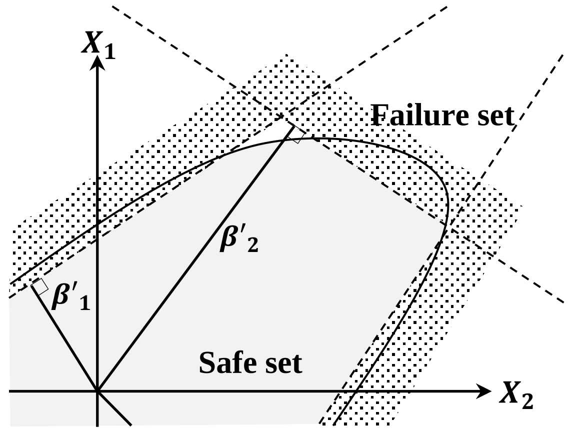


Figure 4.28 SORM approximation [2]

Figure 4.26 depicts a convex safe set. Figure 4.27 introduces the first order reliability method (FORM) approximation. This approximation recognizes that the most essential contributions to the failure probability come from the vicinities of the most central point in a case of linear limit states. It is possible to approximate a convex safe set when the curvature is small by hyperplanes, which remains close to the convex limit state in the vicinity of the most central point. For a given number of hyperplanes chosen to approximate the convex safe set, the linear limit states must be chosen to minimize the probability of failure in order to give the best approximation. F introduces the second order reliability method (SORM) approximation, which is a correction of the FORM approximation when the curvature of the limit state surface is considered. The hyperplanes are shifted toward the inside of the safe set in order to correct the error made by the FORM assumption.

4.8.2 Limit state equation for a single member. In most cases, the member limit state can be written in term of the loading effects on the member S and the member resistance R.

$$f(\boldsymbol{x}) = R(\boldsymbol{x}) - S(\boldsymbol{x})$$

- $\boldsymbol{x}$ is the input variable vector which represents the randomness associated with the model, i.e. the structure geometry and materials properties and the loading.

Usually, when there is no interaction between the loading and the member resistance, $R(\boldsymbol{x})$ is only a function of the member geometry and material characteristics.

$$R(\boldsymbol{x}) = R(\boldsymbol{x_{structure}})$$

- $\boldsymbol{x_{structure}}$ is a family of the input variable vector $\boldsymbol{x}$ which is enough to represent the structure geometry and materials characteristics

Although, the loading effects on the member $S(\boldsymbol{x})$ is a function of the structure and the loading, the linear behavior with regard to the loading of a structure enables to write $S(\boldsymbol{x})$ using two different functions on two different family of the input variable vector $\boldsymbol{x}$.

$$S(\boldsymbol{x}) = S_1(\boldsymbol{x_{structure}}) \cdot x_{loading1} + \cdots + S_q(\boldsymbol{x_{structure}}) \cdot x_{loadingq}$$

$$S(\boldsymbol{x}) = \begin{pmatrix} S_1(\boldsymbol{x_{structure}}) & \cdots & S_q(\boldsymbol{x_{structure}}) \end{pmatrix} \cdot \begin{pmatrix} x_{loading1} & \cdots & x_{loadingq} \end{pmatrix}^T$$

Thus, it is seen that the limit state for a member can be written using function having for domain the set of distinct family of the input variable vector $\boldsymbol{x}$.

4.8.3 Linear limit state approximation. The linear limit state approximation, which is widely used in this method, consists in considering that only the load is a random variable. The random variables describing the structure geometry and materials property are approximated to be constant. These variables are involved in the member resistance and in the loading effects on the member. Recognizing that these variables are actually random variables, they are taken equals to their expected value as follows:

$$R(\boldsymbol{x}) = R(E(\boldsymbol{x}_{\boldsymbol{structure}})) = \begin{pmatrix} E(\boldsymbol{x}_{\boldsymbol{structure1}}) \\ \cdots \\ E(\boldsymbol{x}_{\boldsymbol{structurep}}) \end{pmatrix}$$

And:

$$S(\boldsymbol{x}) = \begin{pmatrix} S_1(\boldsymbol{x}_{\boldsymbol{structure}}) & \cdots & S_q(\boldsymbol{x}_{\boldsymbol{structure}}) \end{pmatrix} \cdot \begin{pmatrix} x_{loading1} \\ \cdots \\ x_{loadingq} \end{pmatrix}$$

$$S(\boldsymbol{x}) = \begin{pmatrix} S_1(E(\boldsymbol{x}_{\boldsymbol{structure}})) & \cdots & S_q(E(\boldsymbol{x}_{\boldsymbol{structure}})) \end{pmatrix} \cdot \begin{pmatrix} x_{loading1} \\ \cdots \\ x_{loadingq} \end{pmatrix}$$

Using this approximation, a linear limit state for a given member is written. It is reasonable to make this approximation because usually the variability associated with the load is larger than the variability associated with the structure. In the case of independent variables, as presented in Fig. 5.29, when a limit state is overpassed, it is usually because a variable had an occurrence far from its expected value, whereas all the other variables had occurrences close to their expected values. Since, the loading is the variable with the largest variability, it is considered that the loading is the only candidate to overpass a limit state while the other variable are taken equals to their expected value. This approximation is depicted by the Fig. 4.29.

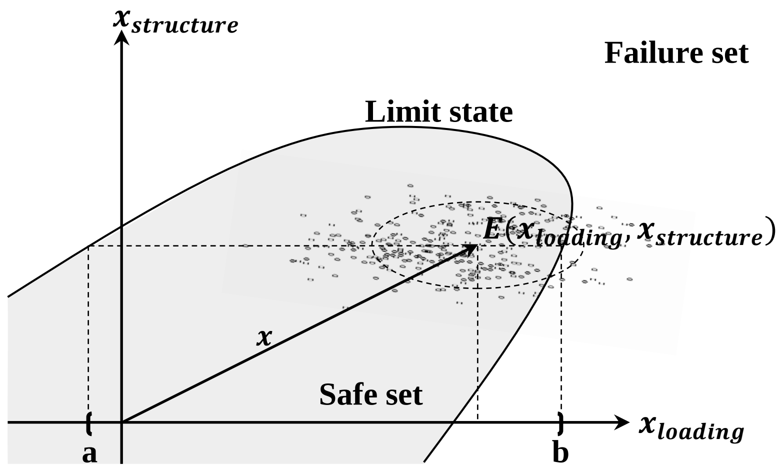


Figure 4.29 Linear limit state approximation and variables scatter

Figure 4.29 depicts a case where the load effect is represented by only one variable and the resistance of the member is represented by a single variable as well. It can be seen that most of the time, when an occurrence of the random vector is in the failure set, the loading variable is far from its expected value; whereas the structure variable is close to its expected value. This motivates the approximation that only the loading variability can be taken into account. Thus, the structure's variables can be replaced by its expected value and a linear limit state is defined in terms of the loading variable. In the case presented by Fig. 5.29, the loading variable must be located between a and b which are the intersection between the limit state and the expected value of the structure's variables.

4.9 Framework of analyses

One of the purposes of this method was to use the classical framework presented in the chapter 3. This framework consists in two principal analyses; namely, a structural analysis and a reliability analysis. The former does not differ in this method but the latter was modified in order to avoid the issues and inconstancies raised by the classical methods.

The inputs of the structural analyses are the structure and the loading and the outputs are the limit state equations for all members. The structural analyses are conducted using the stiffness method. The inputs of the reliability analyses are the limit states equations and the random variables of the load and the outputs are, for each member, the probability of failure, the most central failure point and a new geometry for the structure. The reliability analyses are conducted using the hybrid method introduced in this chapter.

The new geometry of the structure is used as a revised input for a new structural analysis. The knowledge of the most failure point enables to update the random variables. The probabilities of failures are stored for a post analysis (figure 4.30).

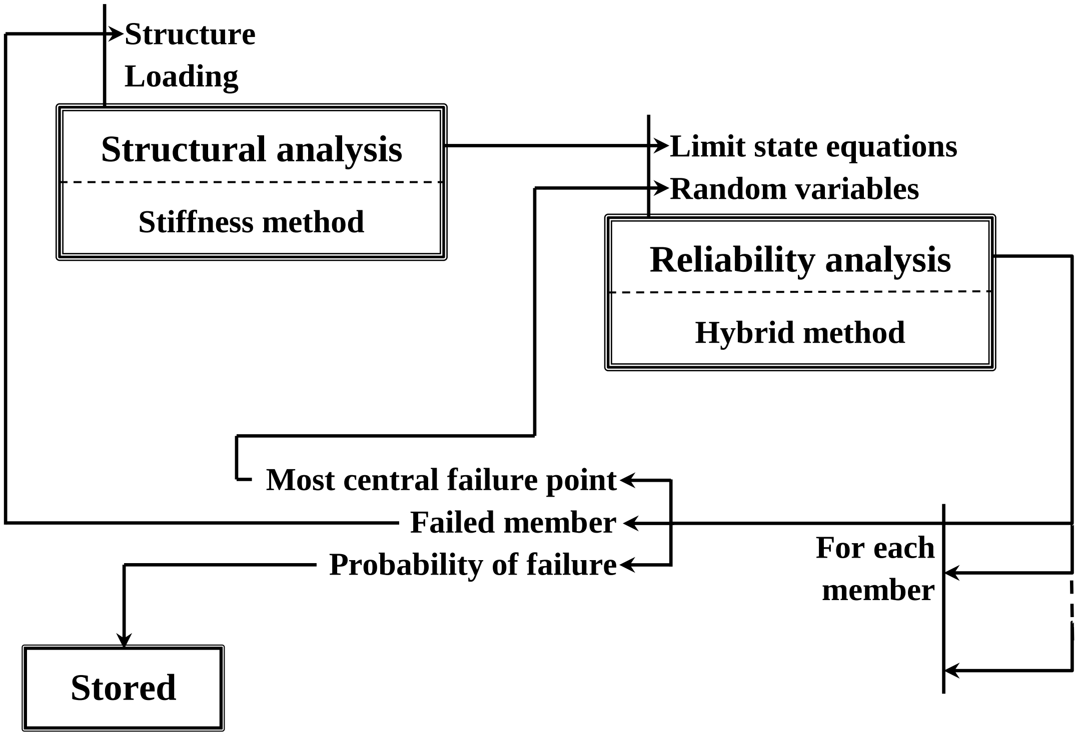


Figure 4.30 Framework of the proposed method

This recursive process must be continued until the formation of a mechanism in the. Such a situation can be detected when the structure stiffness matrix in the structural analysis becomes singular. All along this process, the possible failures paths are computed and, with the computed probability of failure for every member in every scenario, the final probability of failure for the structure can be calculated.

4.10 Probability of failure for the system

4.10.1 The mutually exclusive sets theory. In the classical method, the component reliability is evaluated in terms of R and S which are respectively the resistance of the member and the loading effect on the member. The limit state is represented by the zeros of the function $Z = R - S$ which is assumed to follow a normal distribution in order to compute the probability of failure. Analogously, using the limit state theory introduced in the beginning of this chapter Z could be seen as the surface separating the safe set and the failure set for a given member. The failure would be caused by the occurrence of a vector describing the structure and its loading in the failure set. Pursuing the analogy for an entire structure, in a set composed by the domain of definition of all the structure variables, all the limit states could be represented by a surface separating a safe set and a failure set. In this case, because the failure sets for different members intersect each other, it is said that the limit states are correlated. The classical methods do not deal with the correlation and consider limit state alone in order to compute the probability of failure of a given member. However the correlation is recognized and taken into account when computing the probability of failure for the structure, which is bounded by the two extreme opposite cases where the failure paths are either considered to be entirely correlated or totally independent.

Contrary to the classical methods, the improved method, using the linear state approximation, takes the correlation between member failures into account at the member level. The bounds which are proposed in the hybrid method for the probability of failure of a given member among multiple intersecting member failure sets are close enough and one can choose any of them as a good approximation for the actual member probability of failure. Since, there is no reason to prefer any of the two bounds, the member probability of failure is assumed to be the average of the two bounds. In this method, the member failure probability corresponds to the occurrence of a vector representing the loading in the member failure set which is made mutually exclusive from the other member failure sets by using the load continuity hypothesis. Thus, the member failure sets are a partition of the failure set.

When a member failure occurs, the structure geometry and the random characteristic of the loading are updated. A new failure set is shaped and partitioned with mutually exclusive sets corresponding to the member failure sets. The probability of failure for a second given member is the probability of the updated random variable to occur in the mutually exclusive failure set corresponding to the considered member in the new structure failure set (see figure 4.31).

Finally, the occurrence of a failure path corresponds to the successive occurrences of the successive random input variables in the member failure sets defined on the updated structure failure sets given the history of failure. All these member failure sets are mutually exclusive. The characteristic of the random input variables are defined in function of the scenarios in which they are. It should be noted that these variables are independent from each other.

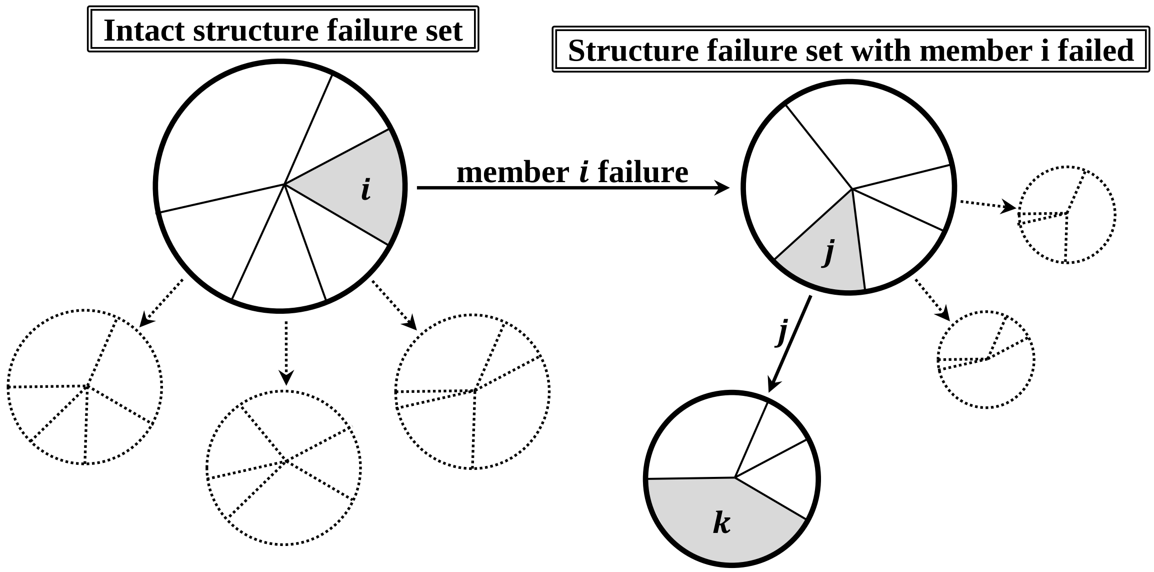


Figure 4.31 Successive mutually exclusive sets

4.10.2 Probability of failure of the structure. The probability of failure can be computed by determining the probability of failure of all failure paths. The occurrence of a failure path corresponds to the occurrence of a given number of independent variables in mutually exclusive failure sets. These independent variables can be taken as a single variable $\boldsymbol{X}$ representing all the outcome of the variables along the failure path. The union of these mutually exclusive sets can be considered as the failure path set $\mathcal{F}$. The probability of occurrence of a failure path is written as follows:

$$P(\boldsymbol{X} \in \mathcal{F}) = P(\boldsymbol{x_1} \in \mathcal{F}_1) \cdot P(\boldsymbol{x_2} \in \mathcal{F}_2 | \boldsymbol{x_1} \in \mathcal{F}_1) \cdot \cdots \cdot P(\boldsymbol{x_n} \in \mathcal{F}_n | \boldsymbol{x_{n-1}} \in \mathcal{F}_{n-1} | \cdots | \boldsymbol{x_1} \in \mathcal{F}_1)$$

Where:

- $\boldsymbol{x_i}$ is the family of $\boldsymbol{X}$ corresponding to the variable defined at the stage i in the failure path.

- $\mathcal{F}_i$ is the subset of $\mathcal{F}$ corresponding to the member failure set at the stage i in the failure path.

The two bounds given in the hybrid method give are close and the average of the two bounds can be used as a good approximation for the probabilities.

Finally, since the failure paths set are composed of mutually exclusive, these sets are also mutually exclusive. Thus, the probability of failure is the sum of the probabilities of occurrence of all the failure paths.

$$P(\mathfrak{F}) = \sum_i P(\boldsymbol{X_i} \in \mathcal{F}_i)$$

Where:

- $\boldsymbol{X_i}$ corresponds to the variable representing all the outcome of the variables along the failure path i.
- $\mathcal{F}_i$ is the failure path set corresponding to the failure path i.

4.11 Concluding remarks

The method reviewed herein successfully overcomes the major shortcomings of the classical method and offers several improvements. These improvements are: (1) The correlation among member failures is taken into account by using mutually exclusive failure event. (2) The load is updated to be more consistent with a real situation thanks to the most failure point. (3) The calculations are faster because unnecessary and inconsistent limit states are removed. (4) Strategies are proposed to take into account material redundancy, i.e. plasticity.

However, the method has its own shortcomings. These are: (1) The limit states must be linear or approximated to be linear. (2) The mutually exclusive events are based on an approximation. (3) The randomness of the structure characteristics is ignored.

CHAPTER 5

ILLUSTRATIVE EXAMPLE

5.1 Study of the intact structure

The geometry of the structure under study is depicted by the figure 5.1. The numbers correspond to the member numbers. During structural analysis, all members are supposed to have the same cross section and material characteristics. The loading of the structure only acts on one joint and is composed of a vertical component x_1 and a horizontal component x_2.

$$\boldsymbol{x} \rightarrow \boldsymbol{E}(\boldsymbol{x}) = \begin{pmatrix} E(x_1) \\ E(x_2) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{x} \rightarrow \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T}) = \begin{pmatrix} 0.20 & 0 \\ 0 & 0.30 \end{pmatrix}$$

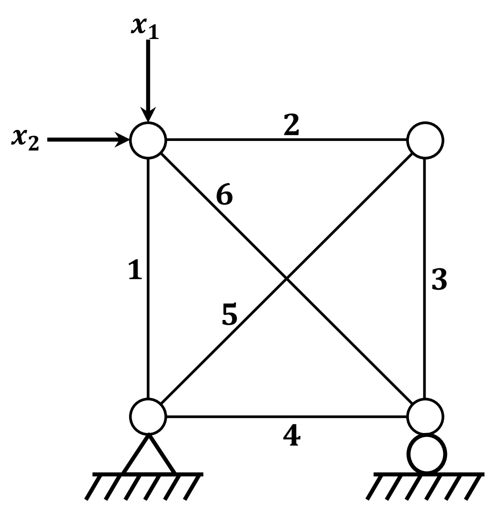


Figure 5.1 Geometry of the structure and member number

Figure 5.2 presents the results of the structural analysis conducted on the structure for a unit load $\boldsymbol{x_1}$ and a unit load $\boldsymbol{x_2}$. The values listed for members correspond to the magnitude of the internal axial forces within the member. A positive number corresponds to a compression: while a negative number corresponds to a tension in the member.

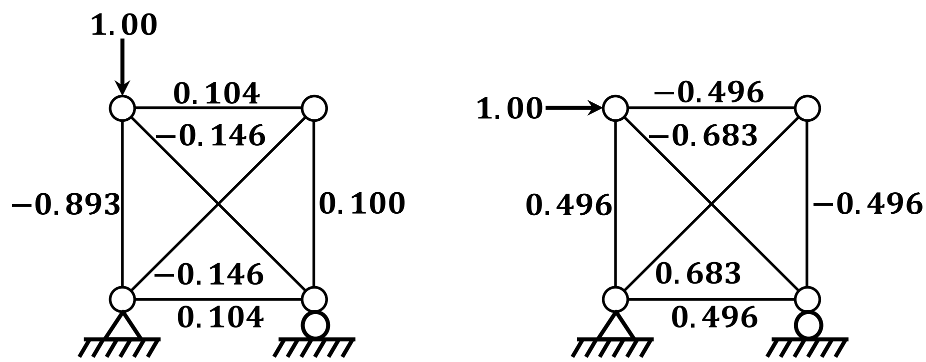


Figure 5.2 Results of the structural analysis for the intact structure

Figure 5.3 represents the member strengths. The members each is designed to resist a loading 1.6 times larger than its expected value, i.e. for design loads $x_1 = 1.6$ and $x_2 = 1.6$. The system was also chosen so that it may slightly increase some members' strengths with different values to provide them different levels of reliability in order to avoid a superposition of limit state hyperplane in the normalized space representation.

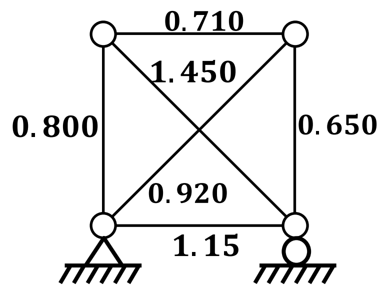


Figure 5.3 Member strengths

With the member strengths known and the structural analysis performed, it is possible to write the member limit states as follows:

$$\begin{cases} -O.893x_1 + 0.496x_2 + 0.800 = 0 \quad (Member\ 1) \\ 0.104x_1 - 0.496x_2 + 0.710 = 0 \quad (Member\ 2) \\ O.100x_1 - 0.496x_2 + 0.650 = 0 \quad (Member\ 3) \\ -0.104x_1 - 0.496x_2 + 1.15 = 0 \quad (Member\ 4) \\ O.146x_1 - 0.683x_2 + 0.920 = 0 \quad (Member\ 5) \\ -O.146x_1 - 0.683x_2 + 1.450 = 0 \quad (Member\ 6) \end{cases}$$

It should be noted that a unique mode of failure is chosen for each member. This mode of failure is either a compression failure or a tension failure depending on the sign of the force within the member for the expected values of x_1 and x_2. The limit state can be written in the normalized space using the following transformation.

$$y_1 = \frac{x_1 - 1.00}{0.20} \qquad\qquad 0.20y_1 + 1.00 = x_1$$
$$\Leftrightarrow$$
$$y_2 = \frac{x_1 - 1.00}{0.30} \qquad\qquad 0.30y_2 + 1.00 = x_2$$

And, in the normalized space:

$$\begin{cases} -0.179y_1 + 0.149y_2 + 0.403 = 0 \quad (Member\ 1) \\ 0.0208y_1 - 0.149y_2 + 0.318 = 0 \quad (Member\ 2) \\ O.0200y_1 - 0.149y_2 + 0.134 = 0 \quad (Member\ 3) \\ -0.0208y_1 - 0.149y_2 + 0.550 = 0 \quad (Member\ 4) \\ O.0292y_1 - 0.207y_2 + 0.383 = 0 \quad (Member\ 5) \\ -O.0292y_1 - 0.205y_2 + 0.621 = 0 \quad (Member\ 6) \end{cases}$$

Figure 5.4 is the representation of these limit states in the normalized space.

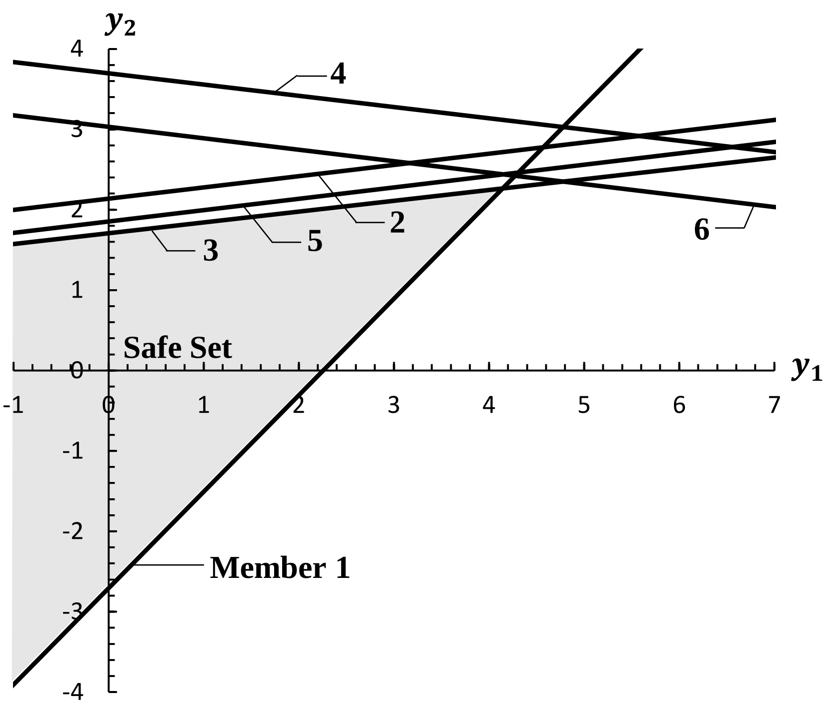


Figure 5.4 Limit states in the normalized space for the intact structure

As was discussed in the chapter 5, from the first representation of all limit states in the normalized state, it can be seen that some limit states are not consistent with reality because of the continuous evolution of the load. In that case, the limit states of members

2, 4, 5 and 6 do not represent possible failures and should be removed. Figure 5.5 is a consistent representation of the possible failures within the structure.

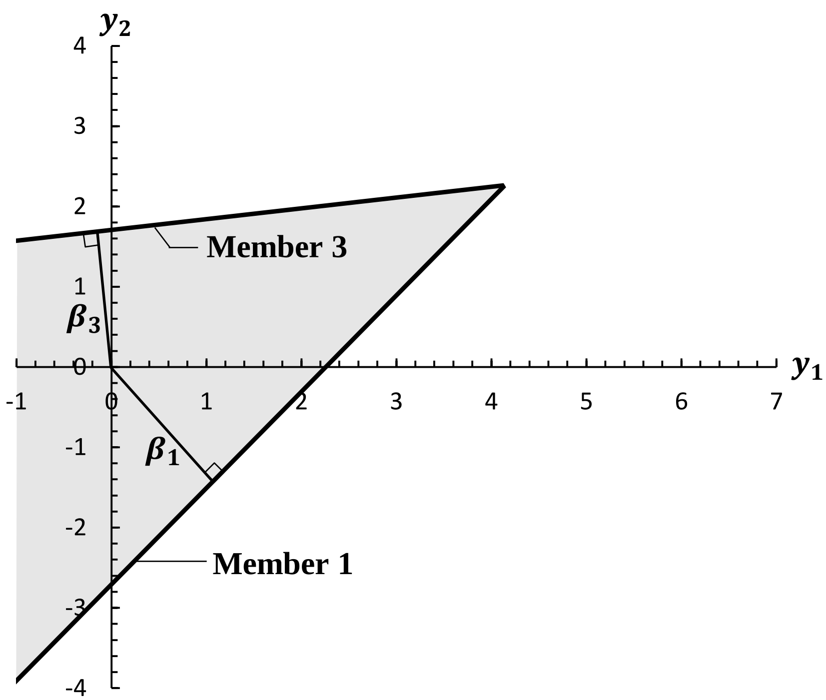


Figure 5.5 Consistent limit states for the intact structure

It is possible, by using the β index and by integrating on the hyperplane, to calculate the probability of failure for the member 1 and for the member 3. The results are summarized in the Table 5.1.

Table 5.1 Probability of member failure for the intact structure

Member	β	$\Phi(-\beta)$	$\int_{y \in \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$	P_f
1	1.61	0.0537	1	0.0537
3	1.41	0.0793	1	0.0793

As shown in Fig. 5.5, among the 6 possible failures only two must be considered. In order to update the characteristics of the loading for further analyses, the most central failure points for the failure of the member 1 and for the failure of the member 3 must be computed (see Table 5.2).

Table 5.2 Most central failure point for the intact structure

Member	In the normalized space	In the loading space
1	$y_1 = 1.1$	$x_1 = 1.22$
	$y_2 = -1.4$	$x_2 = 0.58$
3	$y_1 = -0.1$	$x_1 = 0.98$
	$y_2 = 1.7$	$x_2 = 1.51$

5.2 Study of the structure with the member 1 failed

The loading of the structure is composed of three forces. The two first forces are the random loadings which have the following updated characteristics:

$$\boldsymbol{x} \rightarrow \boldsymbol{E}(\boldsymbol{x}) = \begin{pmatrix} E(x_1) \\ E(x_2) \end{pmatrix} = \begin{pmatrix} 1.22 \\ 0.58 \end{pmatrix}$$

$$\boldsymbol{x} \rightarrow \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{T}}) = \begin{pmatrix} 0.20 & 0 \\ 0 & 0.30 \end{pmatrix}$$

And the last force corresponds to the plastic force within the member 1 which has a value of 0.800. Figure 5.6 presents the results of the structural analysis conducted on the structure for a unit load x_1 and a unit load x_2. Figure 5.7 represents the forces in the structures caused by the member 1 plastic force.

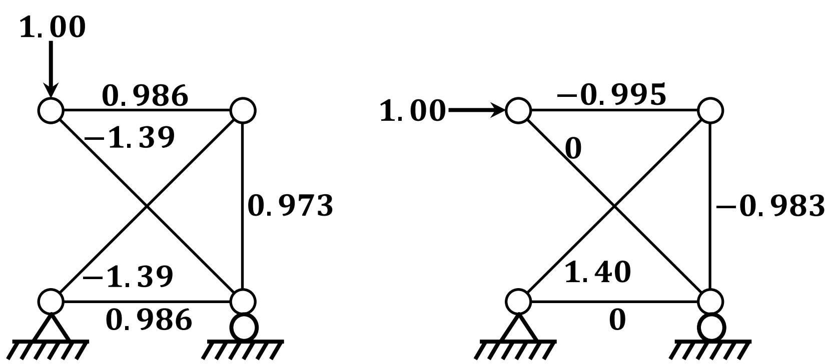


Figure 5.6 Results of the structural analysis for the structure with member 1 failed

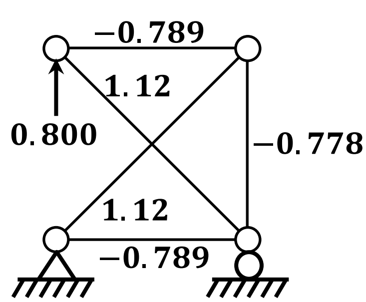


Figure 5.7 Forces in members caused by the member 1 plastic force

At this stage, the member strengths are still the same; and upon structural analysis, it is possible to write the member limit states as follows:

$$\begin{cases} 0.986x_1 - 0.995x_2 - 0.0790 = 0 \quad (Member\ 2) \\ 0.973x_1 - 0.983x_2 - 0.118 = 0 \quad (Member\ 3) \\ -0.986x_1 + 1.94 = 0 \quad (Member\ 4) \\ -1.39x_1 + 2.57 = 0 \quad (Member\ 5) \\ 1.39x_1 - 1.40x_2 - 0.200 = 0 \quad (Member\ 6) \end{cases}$$

These limit states can be written in the normalized space using the following transformation:

$$y_1 = \frac{x_1 - 1.22}{0.20} \qquad \qquad 0.20y_1 + 1.22 = x_1$$
$$\Leftrightarrow$$
$$y_2 = \frac{x_1 - 0.58}{0.30} \qquad \qquad 0.30y_2 + 0.58 = x_2$$

And, in the normalized space,

$$\begin{cases} 0.197y_1 - 0.298y_2 + 0.547 = 0 \quad (Member\ 2) \\ 0.195y_1 - 0.295y_2 + 0.499 = 0 \quad (Member\ 3) \\ -0.197y_1 + 0.737 = 0 \quad (Member\ 4) \\ 0.278y_1 - 0.420y_2 + 0.683 = 0 \quad (Member\ 5) \\ -0.278y_1 + 0.874 = 0 \quad (Member\ 6) \end{cases}$$

Figure 5.8 is a representation of these limit states in the normalized space.

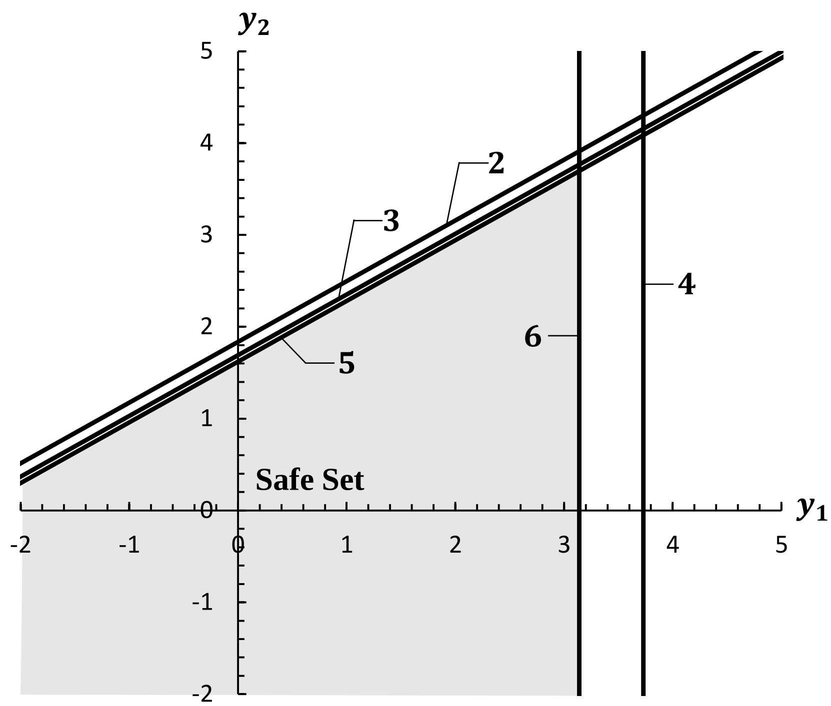


Figure 5.8 Limit states in the normalized space for the structure with member 1 failed

From Fig. 5.8, it is possible to detect which limit states are not consistent and to plot a consistent representation of the situation (see Fig. 5.9).

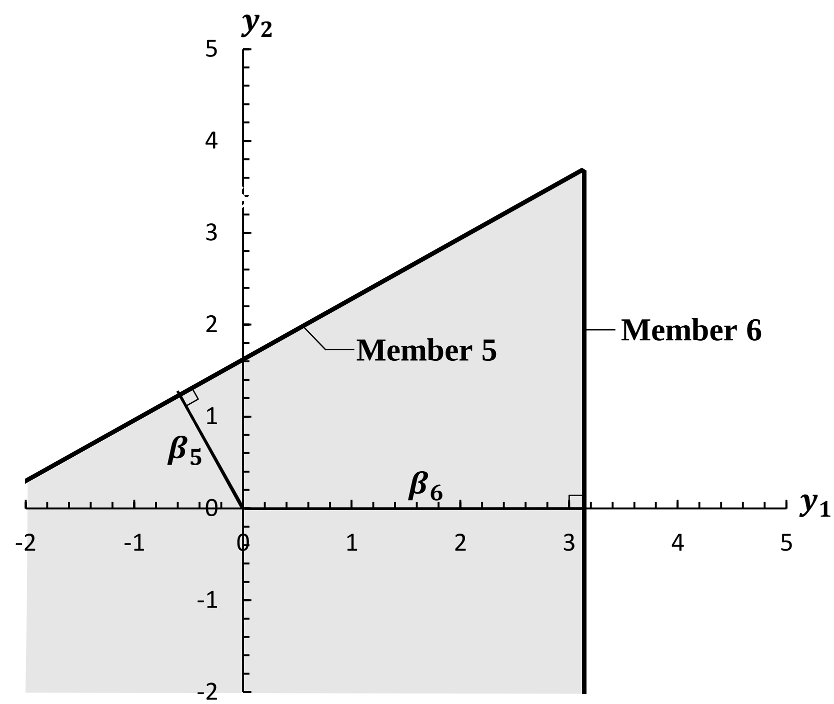


Figure 5.9 Consistent limit states for the structure with member 1 failed

Again, by using the β index and by integrating on the hyperplane, the probability of failure for the member 5 and for the member 6 are computed. The results are summarized in Table 5.3.

Table 5.3 Probability of member failure for the structure with member 1 failed

Member	β	$\Phi(-\beta)$	$\int_{y \in \mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$	P_f
5	1.26	0.104	1	0.104
6	3.14	0.000840	1	0.000840

Since the failure of a second member in the structure induces a mechanism, the structure is considered to be failed after the failure of the member 5 or the member 6 given that the member 1 has already failed. The end of the failure path is then reached.

5.3 Study of the structure with the member 3 failed

The loading of the structure is composed of three forces. The two first forces are the random loading which has the following updated characteristics:

$$\boldsymbol{x} \rightarrow \boldsymbol{E}(\boldsymbol{x}) = \begin{pmatrix} E(x_1) \\ E(x_2) \end{pmatrix} = \begin{pmatrix} 0.98 \\ 1.51 \end{pmatrix}$$

$$\boldsymbol{x} \rightarrow \boldsymbol{Cov}(\boldsymbol{x}, \boldsymbol{x^T}) = \begin{pmatrix} 0.20 & 0 \\ 0 & 0.30 \end{pmatrix}$$

And the last force corresponds to the plastic force within the member 3, which has a value of 0.650. Figure 5.10 presents the results of the structural analysis conducted on the structure for a unit load x_1 and a unit load x_2. Figure 5.11 represents the forces in the structures caused by the member 3 plastic force.

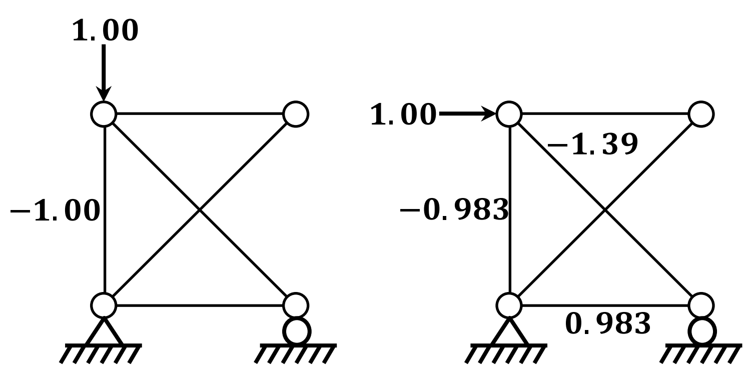


Figure 5.10 Results of the structural analysis for the structure with member 3 failed

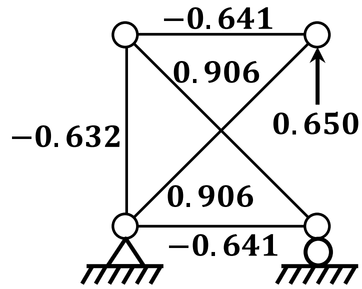


Figure 5.11 Forces in members caused by the member 3 plastic force

The member strengths are still the same; and upon structural analysis, it is possible to write the member limit states as follows:

$$\begin{cases} -1.00x_1 + 0.983x_2 + 0.168 = 0 \quad (Member\ 1) \\ -0.983x_2 + 1.79 = 0 \quad (Member\ 4) \\ -1.39x_2 + 2.35 = 0 \quad (Member\ 6) \end{cases}$$

Again, these limit states can be written in the normalized space using the following transformation:

$$\begin{matrix} y_1 = \dfrac{x_1 - 0.98}{0.20} \\ y_2 = \dfrac{x_1 - 1.51}{0.30} \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} 0.20y_1 + 0.98 = x_1 \\ 0.30y_2 + 1.51 = x_2 \end{matrix}$$

And, in the normalized space,

$$\begin{cases} -0.2y_1 + 0.295y_2 + 0.672 = 0 \quad (Member\ 1) \\ -0.295y_2 + 0.307 = 0 \quad (Member\ 4) \\ -0.417y_2 + 0.256 = 0 \quad (Member\ 6) \end{cases}$$

Figure 5.12 is a representation of these limit states in the normalized space.

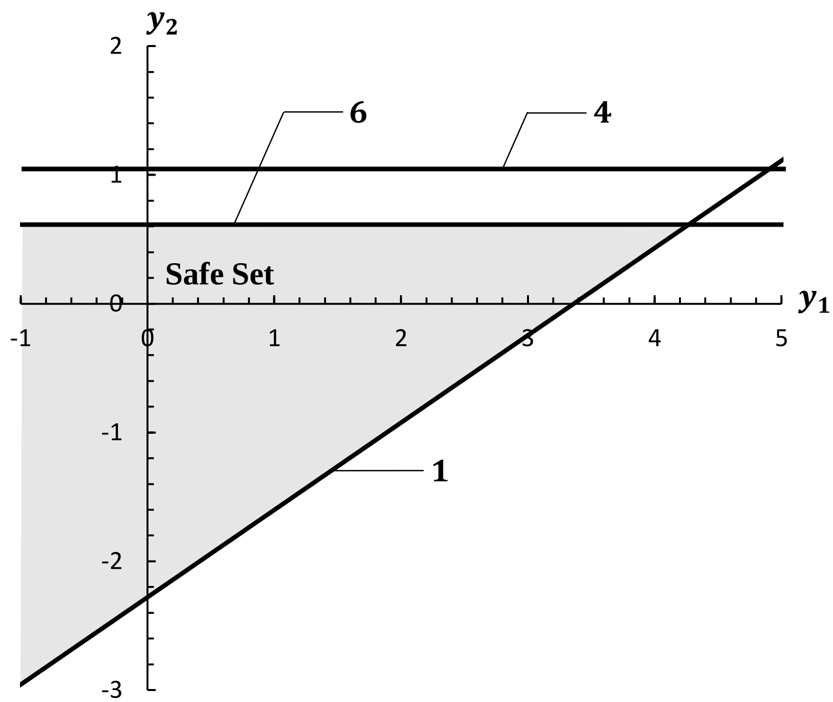


Figure 5.12 Limit states in the normalized space for the structure with member 3 failed

Again, using Fig. 5.12, it is possible to detect which limit states are not consistent and to plot a consistent representation of the situation (see figure 5.13).

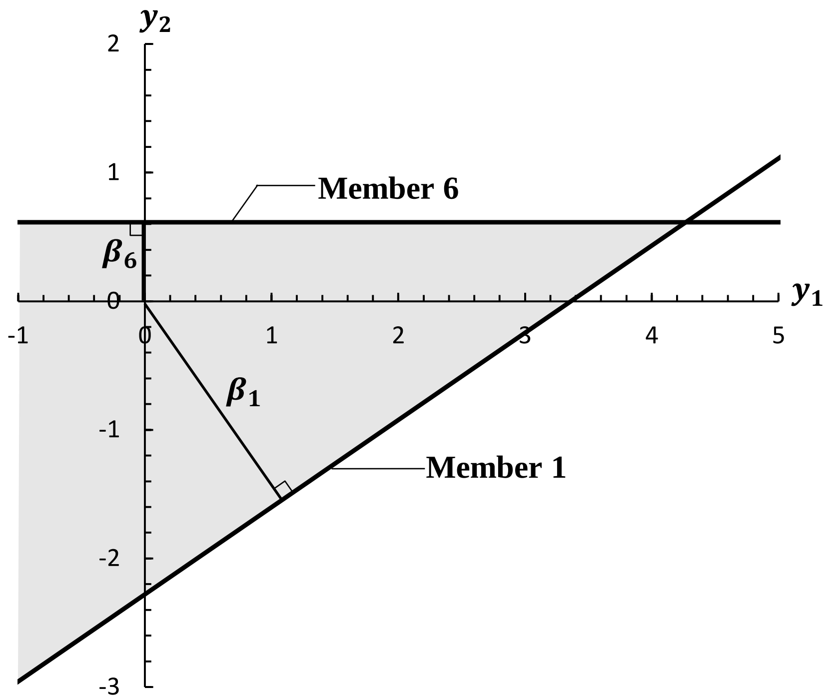


Figure 5.13 Consistent limit states for the structure with member 3 failed

Again, by using the β index and by integrating on the hyperplane, the probability of failure for the member 1 and for the member 6 are calculated. The results are summarized in Table 5.4.

Table 5.4 Probability of member failure for the structure with member 3 failed

Member	β	$\Phi(-\beta)$	$\int_{y\,\epsilon\,\mathcal{R}} \left(\frac{1}{\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}r^2} dr$	P_f
1	1.91	0.0281	1	0.0281
6	0.61	0.271	1	0.271

Since the failure of a second member in the structure induces a mechanism, the structure is considered to have failed after the failure of the member 1 or the member 6 given that the member 3 has already failed. The end of the failure path is therefore reached.

5.4 Structure probability of failure

The structure probability of failure is based on the occurrence of failure paths. The probability of occurrence of a failure path is taken as the multiplication of the probability of failure of all members involved in the failure path. Table 5.5 gives the probability of occurrence of the possible failure paths.

Table 5.5 Probability of occurrence of failure paths

Failure path	Probability of occurrence
(1,5)	$5.58 \cdot 10^{-3}$
(1,6)	$4.51 \cdot 10^{-5}$
(3,1)	$2.23 \cdot 10^{-3}$
(3,6)	$2.15 \cdot 10^{-2}$

Finally, the structure probability of failure is taken as the sum of the probability of occurrence of all failure paths.

$$P_f = 5.58 \cdot 10^{-3} + 4.51 \cdot 10^{-5} + 2.23 \cdot 10^{-3} + 2.15 \cdot 10^{-2} = 2.93 \cdot 10^{-2}$$

The failure paths can be represented in a tree as depicted by Fig. 5.14.

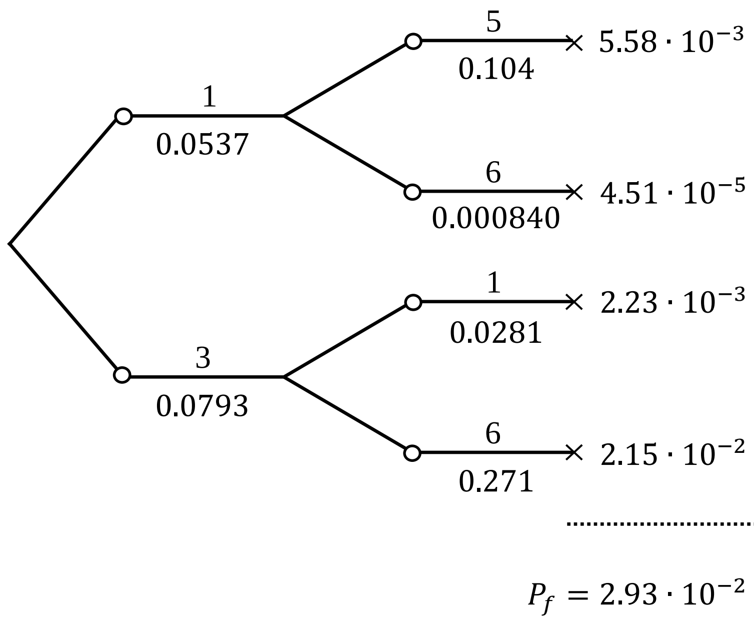


Figure 5.14 Failure paths tree for the example structure

CHAPTER 6

SUMMARY AND CONCLUSIONS

6.1 Summary

This thesis presents an overview of an improved method that can be used for the probabilistic evaluation of the redundancy in indeterminate trusses. The improved method offers advantages over classics method by considering:

- An approximation to make the events of member failures mutually exclusive.
- Geometric calculations to determine reliability indices and conditional reliability indices in order to establish closer bounds for the failure probability of individual structural members.
- System's failure probability using the assumption and procedures outlined in (1) and (2) above.
- Further extending the method beyond geometrical redundancy by using material redundancy, and plasticity models commonly used in the structural analyses.

The details of the improved methods, as provided in literature, are provided in the thesis. A numerical example is presented to demonstrate the sequence of failure in a truss system and how various improvements can be used to arrive at the probability of system failure for a simple redundant truss. The purpose of this example is only to demonstrate the methods reviewed herein without any specific comments in terms of any design recommendations. The following are the main conclusions of the study:

- The improved method for probabilistic evaluation of structural redundancy can be applied to redundant systems by considering geometrical and material redundancies.
- Certain approximation regarding the failure limit states and formulation of probability can be made to make the method efficient. Among these include removal of members with significantly low probability of failure in individual failure paths.
- The incorporation of material redundancy and geometrical properties as well as consideration of possible failures among members are expected to provide more accurate results than those using classical methods.

6.2 Recommendations for further studies

The following studies are proposed for the extension of this research.

- Conduct the analysis for several real trusses to investigate their redundancy though the improved method.
- Develop software to automatically identify failure paths in reducing the computation effort.
- Apply classical methods for these trusses to develop a baseline for comparing results from the improved method.
- Suggest specific case studies where the load can change during member failures, as indicated in the improved method.

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